

BIVARIANT K -THEORY WITH \mathbb{R}/\mathbb{Z} -COEFFICIENTS AND RHO CLASSES OF UNITARY REPRESENTATIONS

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ABSTRACT. We construct equivariant KK -theory with coefficients in \mathbb{R} and \mathbb{R}/\mathbb{Z} as suitable inductive limits over II_1 -factors. We show that the Kasparov product, together with its usual functorial properties, extends to KK -theory with real coefficients.

Let Γ be a group. We define a Γ -algebra A to be *K -theoretically free and proper* (KFP) if the group trace tr of Γ acts as the unit element in $KK_{\mathbb{R}}^{\Gamma}(A, A)$. We show that free and proper Γ -algebras (in the sense of Kasparov) have the (KFP) property. Moreover, if Γ is torsion free and satisfies the KK^{Γ} -form of the Baum-Connes conjecture, then every Γ -algebra satisfies (KFP).

If $\alpha : \Gamma \rightarrow U_n$ is a unitary representation and A satisfies property (KFP), we construct in a canonical way a rho class $\rho_{\alpha}^A \in KK_{\mathbb{R}/\mathbb{Z}}^{1, \Gamma}(A, A)$. This construction generalizes the Atiyah-Patodi-Singer K -theory class with \mathbb{R}/\mathbb{Z} coefficients associated to α .

INTRODUCTION

Let V be a closed manifold with fundamental group Γ . The symmetric index of an elliptic pseudodifferential operator D on V is an element of the K -theory of the group C^* -algebra $\text{ind}_{\Gamma}(D) \in K_0(C^*\Gamma)$.

Atiyah's L^2 -index theorem for covering spaces ([2]) expresses the triviality of the group trace on this element: it states that the image of the index class $\text{ind}_{\Gamma}(D)$ by the trivial representation of Γ , which is the ordinary index of D , coincides with the image of $\text{ind}_{\Gamma}(D)$ by the group trace of Γ , *i.e.* the von Neumann index of the associated Γ -invariant operator acting on the universal cover \tilde{V} .

This property of the group trace in Atiyah's theorem is equivalent to the fact that the Mishchenko bundle with fibre a II_1 -factor can be trivialized. This plays a crucial role in the construction of secondary invariants.

Let $\alpha : \Gamma \rightarrow U_n$ be a unitary representation. Atiyah, Patodi and Singer constructed a class $[\alpha]_{\text{APS}}$ in the K -theory of V with \mathbb{R}/\mathbb{Z} -coefficients in a way that the pairing of $[\alpha]_{\text{APS}}$ with the K -homology class $[D]$ is equal in \mathbb{R}/\mathbb{Z} to the reduced rho invariant [3, 4].

We showed in [1], to which we refer the reader for a more comprehensive literature, that the class $[\alpha]_{\text{APS}}$ can be given by a purely K -theoretical construction, using von Neumann algebras. In this description, $[\alpha]_{\text{APS}}$ is a relative class, represented by two bundles which become isomorphic when twisted by a bundle with fiber a II_1 -factor. In a sense we showed that $[\alpha]_{\text{APS}}$ can be constructed as a secondary invariant built as a consequence of the triviality of the trace in Atiyah's L^2 -index theorem.

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In the present paper, we examine general cases for which the same triviality of the trace occurs and associate to them secondary invariants in an appropriate KK -theory with \mathbb{R}/\mathbb{Z} coefficients.

To do so, we first have to give a general definition of KK -theory with \mathbb{R} and \mathbb{R}/\mathbb{Z} coefficients.

It is quite clear that the KK -theory with real coefficients should naturally be defined using II_1 -factors; in other words, one is tempted to put $KK_{\mathbb{R}}(A, B) = KK(A, B \otimes M)$ where M is a II_1 -factor. We immediately fall into the question: which II_1 -factor should be chosen? If A and B are in the bootstrap category, the group $KK(A, B \otimes M)$ is equal to $\text{Hom}(K_*(A) \rightarrow K_*(B) \otimes \mathbb{R})$ whence does not depend on the II_1 -factor M . But in general, this group depends (a priori) on M . There is of course a “minimal choice” for M , namely the hyperfinite II_1 -factor R , but we need here factors big enough to contain the group von Neumann algebra of our Γ : then it is not reasonable in this discussion to assume Γ to be amenable, since the typical Γ is a dense subgroup of U_n . On the opposite, Ozawa ([22]) proved that there is no “maximal choice” for M .

Our solution is to define $KK_{\mathbb{R}}(A, B)$ as the *inductive limit of $KK(A, B \otimes M)$ over all II_1 -factors M* . It is not too difficult to give a sense to this inductive limit. One extends also very easily the Kasparov product with all its functorial properties and associativity to the $KK_{\mathbb{R}}$ -theory (using von Neumann tensor products).

We then define $KK_{\mathbb{R}/\mathbb{Z}}$ as the inductive limit of $KK(A, B \otimes \text{Cone}(\mathbb{C} \rightarrow M))$. The mapping cone exact sequence yields the Bockstein change of coefficients sequence.

In the same way, one defines all the equivariant KK -theories with \mathbb{R} and \mathbb{R}/\mathbb{Z} coefficients, letting the group (or more generally a group-like object) act trivially on the II_1 -factors.

A tracial state τ on a C^* -algebra D may be thought of as a generalized morphism to a II_1 -factor: in fact, there is a II_1 -factor M and a tracial morphism $D \rightarrow M$, so that we obtain a class $[\tau]$ in $KK_{\mathbb{R}}(D, \mathbb{C})$. Using Kasparov product, we then obtain morphisms $\tau_* : KK(A, B \otimes D) \rightarrow KK_{\mathbb{R}}(A, B)$ for every C^* -algebras A and B .

Let now Γ be a group. A tracial state on $C^*\Gamma$ defines an element in $KK_{\mathbb{R}}(C^*\Gamma, \mathbb{C})$. This group bears a ring structure using the coproduct of Γ , and is actually equal to the equivariant group $KK_{\mathbb{R}}^{\Gamma}(\mathbb{C}, \mathbb{C})$.

Denote by \mathbf{tr} the group trace of Γ and $[\mathbf{tr}]^{\Gamma}$ its class, which is an idempotent of the ring $KK_{\mathbb{R}}^{\Gamma}(\mathbb{C}, \mathbb{C})$. As all the Kasparov groups $KK_{\mathbb{R}}^{\Gamma}(A, B)$ are modules over this ring, a natural question is: what is the image of this idempotent on these $KK_{\mathbb{R}}^{\Gamma}(\mathbb{C}, \mathbb{C})$ modules?

We will be interested on those C^* -algebras A endowed with an action of Γ (in short, such an A is called a Γ -algebra), for which the element $[\mathbf{tr}]^{\Gamma}$ acts as a unit element in $KK_{\mathbb{R}}^{\Gamma}(A, A)$. This means that $1_A^{\Gamma} \otimes_{\mathbb{C}} [\mathbf{tr}]^{\Gamma} = 1_{A, \mathbb{R}}^{\Gamma} \in KK_{\mathbb{R}}^{\Gamma}(A, A)$. When this holds, we say that A satisfies property (KFP).

This definition is inspired by the commutative case where the deck group Γ of a Galois covering \tilde{V} of a closed manifold V acts on $C_0(\tilde{V})$. Atiyah’s theorem in [2] can be interpreted as stating that the Γ -algebra $C_0(\tilde{V})$ satisfies property (KFP).

In this commutative case, property (KFP) comes from the fact that the action of Γ on \tilde{V} is free and proper. This is why algebras satisfying this condition are thought of as being in a sense K -theoretically free and proper.

Let $[\alpha]^{\Gamma} \in KK^{\Gamma}(\mathbb{C}, \mathbb{C})$ be the class of a unitary representation $\alpha : \Gamma \rightarrow U_n(\mathbb{C})$ in the Kasparov representation ring. Since the regular representation of Γ absorbs, we have $[\alpha]^{\Gamma} \otimes [\mathbf{tr}]^{\Gamma} = n \cdot [\mathbf{tr}]^{\Gamma}$. If A is a Γ -algebra with property (KFP), then the difference $1_A^{\Gamma} \otimes [\alpha]^{\Gamma} - n 1_A^{\Gamma} \in$

$KK^\Gamma(A, A)$ vanishes in $KK_{\mathbb{R}}^\Gamma(A, A)$, and thus has a lift under the boundary map

$$\partial : KK_{\mathbb{R}/\mathbb{Z}}^{1,\Gamma}(A, A) \longrightarrow KK^\Gamma(A, A)$$

of the Bockstein sequence. Our main construction, Theorem 2.7, is in fact a transgression of $[\alpha]^\Gamma$, *i.e.* a canonical lift $\rho_\alpha^A \in KK_{\mathbb{R}/\mathbb{Z}}^{1,\Gamma}(A, A)$ that we call the *rho class* associated to A and α . It generalizes the APS class and it is additive with respect to direct sum of representations; its behavior with respect to tensor products is also easy to describe. Furthermore ρ_α^A is functorial with respect to the algebra A and, more generally, with respect to the action of the KK^Γ -groups by Kasparov product.

To complete the construction, it is then natural to exhibit classes of algebras satisfying property (KFP).

- The first example comes from the *Mishchenko bundle*, *i.e.* the already mentioned cocompact regular covering space $\tilde{V} : C_0(\tilde{V})$ satisfies property (KFP). In this case, we compare the rho class $\rho_\alpha^{C_0(\tilde{V})}$ with the element $[\alpha]_{APS}$.
- Based on the example of the Mishchenko bundle, we prove that free and proper Γ -algebras in the sense of Kasparov satisfy property (KFP).
- Moreover, property (KFP) is obviously invariant under KK^Γ -(sub)equivalence. It follows that if Γ is torsion free and satisfies the KK^Γ -form of the Baum-Connes conjecture, then every Γ -algebra satisfies property (KFP).
- If Γ is torsion free and has a γ element in the sense of Kasparov, then we can construct in this way the γ part of ρ_α^C , and therefore of ρ_α^A for any Γ -algebra A .
- Using a result of Guentner-Higson-Weinberger ([15]), we moreover deduce a construction of ρ_α^C whenever $\alpha(\Gamma)$ has no torsion.

Finally, we state a weakening of property (KFP), called the weak (KFP) property, saying that the image of $[\mathbf{tr}]$ by Kasparov's descent morphism j_Γ in $KK_{\mathbb{R}}(A \rtimes \Gamma, A \rtimes \Gamma)$ is the unit element. Under this condition, we construct a weaker rho class $\hat{\rho}_\alpha^A \in KK_{\mathbb{R}/\mathbb{Z}}^1(A \rtimes \Gamma, A \rtimes \Gamma)$. We show that these two constructions are related: if A satisfies property (KFP), then $\hat{\rho}_\alpha^A = j_\Gamma(\rho_\alpha^A)$.

Here is a summary of our paper:

- In the first section, we construct the KK -theory with coefficients \mathbb{R} and \mathbb{R}/\mathbb{Z} . We also discuss a few other K -theoretic constructions as the torus algebra of a pair of maps $\varphi_i : A \rightarrow B$ and the corresponding six term exact sequence in (equivariant) KK -theory.
- In the second section we introduce property (KFP) and construct the rho element associated to a finite dimensional unitary representation.
- In the third section we discuss examples of Γ -algebras with property (KFP).
- Finally, in section 4, we introduce a weakening of property (KFP) and construct the corresponding weak rho class.

1. K -THEORETIC CONSTRUCTIONS

1.1. Some conventions. We will freely use Kasparov's KK -groups and notation from [18, 19].

Let us fix a few conventions that will be used throughout the text:

- In what follows, by trace on a C^* -algebra we will mean a finite (positive) trace.

- All traces on von Neumann algebras that we consider, and all morphisms are assumed to be normal.
- The suspension of a C^* -algebra A is $SA = C_0((0, 1), A)$.
- If A is a C^* -algebra, we denote by 1_A the unit element of the ring $KK(A, A)$. If A is a Γ -algebra, the unit element of the ring $KK^\Gamma(A, A)$ will be denoted by 1_A^Γ . If $u : A \rightarrow B$ is an equivariant morphism, we will denote by $[u]^\Gamma \in KK^\Gamma(A, B)$ its class: $[u]^\Gamma = u^*(1_B^\Gamma) = u_*(1_A^\Gamma)$.
- More generally, if E is an equivariant Hilbert B -module endowed with a left action of A through an equivariant morphism $A \rightarrow \mathcal{K}(E)$, we will denote by $[E]^\Gamma$ the corresponding class in $KK^\Gamma(A, B)$.
- In what follows, \otimes means minimal tensor product. Note however that we could do the construction below using maximal tensor products (and normal tensor products wherever von Neumann algebras are involved) as well.

However, if Γ is a locally compact group, $C^*\Gamma$ will denote the *full* group C^* -algebra of Γ .

- If $u \in KK^\Gamma(A_1, B_1)$ and $v \in KK^\Gamma(A_2, B_2)$, we will denote by $u \otimes v \in KK^\Gamma(A_1 \otimes A_2, B_1 \otimes B_2)$ their external KK -product.
- In particular, if $x \in KK^\Gamma(A, B)$ and D is a Γ -algebra $1_D^\Gamma \otimes x \in KK^\Gamma(D \otimes A, D \otimes B)$ is the element noted $\tau_D(x)$ in [18].

Remark 1.1. In this paper, we deal with KK -theory in connection with von Neumann algebras. A von Neumann algebra is (almost) never a separable C^* -algebra.

On the other hand, recall from [29, Remark 3.2] that if A and B are C^* -algebras with A separable then $KK(A, B)$ is the direct limit over the separable subalgebras of B . One then should define $KK(A, B)$ with A not separable as the projective limit over all separable subalgebras of A . Kasparov's KK -group maps to this new $KK(A, B)$ and, with this definition, the Kasparov product $\otimes_D : KK(A, D) \times KK(D, B) \rightarrow KK(A, B)$ is defined for any triple of C^* -algebras with no assumptions of separability of any kind, as well as the more general $\otimes_D : KK(A_1, B_1 \otimes D) \times KK(D \otimes A_2, B_2) \rightarrow KK(A_1 \otimes A_2, B_1 \otimes B_2)$.

1.2. Torus algebra. Let A, B be C^* -algebras and $\varphi_0, \varphi_1 : A \rightarrow B$ two $*$ -homomorphisms. Define the corresponding *torus algebra* to be

$$\mathcal{T}(\varphi_0, \varphi_1) = \{(a, f) \in A \times B[0, 1]; f(0) = \varphi_0(a) \text{ and } f(1) = \varphi_1(a)\}.$$

We will use the following straightforward functorial properties:

- (1) if $\varphi_0, \varphi_1 : A \rightarrow B$ then $\mathcal{T}(\varphi_0, \varphi_1) \otimes D = \mathcal{T}(\varphi_0 \otimes \text{id}_D, \varphi_1 \otimes \text{id}_D)$.
- (2) Every $f : D \rightarrow A$ induces $f^* : \mathcal{T}(\varphi_0 \circ f, \varphi_1 \circ f) \rightarrow \mathcal{T}(\varphi_0, \varphi_1)$.
- (3) Every $f : B \rightarrow D$ induces $f_* : \mathcal{T}(\varphi_0, \varphi_1) \rightarrow \mathcal{T}(f \circ \varphi_0, f \circ \varphi_1)$.
- (4) Let $\varphi_0, \varphi_1, \psi_0, \psi_1 : A \rightarrow B$; if φ_i is homotopic to ψ_i , the corresponding torus algebras $\mathcal{T}(\varphi_0, \varphi_1)$ and $\mathcal{T}(\psi_0, \psi_1)$ are homotopy equivalent. More precisely, if $\Phi_0, \Phi_1 : A \rightarrow B[0, 1]$ are morphisms joining φ_0 to ψ_0 and φ_1 to ψ_1 (i.e. such that $(\Phi_i(a))(0) = \varphi_i(a)$ and $(\Phi_i(a))(1) = \psi_i(a)$ for all $a \in A$), we construct a homotopy equivalence $\mathcal{T}(\varphi_0, \varphi_1) \rightarrow \mathcal{T}(\psi_0, \psi_1)$ mapping (a, f) to (a, g) , where

$$g(t) = \begin{cases} \Phi_0(1 - 3t) & \text{if } 0 \leq t \leq 1/3 \\ f(3t - 1) & \text{if } 1/3 \leq t \leq 2/3 \\ \Phi_1(3t - 2) & \text{if } 2/3 \leq t \leq 1 \end{cases}$$

We have an exact sequence $0 \longrightarrow SB \xrightarrow{h} \mathcal{T}(\varphi_0, \varphi_1) \xrightarrow{p} A \longrightarrow 0$ (with completely positive lifting), where $p : \mathcal{T}(\varphi_0, \varphi_1) \rightarrow A$ is the morphism $(a, f) \mapsto a$ and $h : SB \rightarrow \mathcal{T}(\varphi_0, \varphi_1)$ is the map $f \mapsto (0, f)$.

The corresponding connecting map is $[\varphi_0] - [\varphi_1]$. We then have:

Proposition 1.2. *Let Γ be a discrete group, and let A, B, C, D be Γ -algebras. Let $\varphi_0, \varphi_1 : A \rightarrow B$ be Γ -equivariant morphisms. We have six term exact sequences*

$$\begin{array}{ccccc} KK^\Gamma(C, D \otimes SB) & \xrightarrow{h_*} & KK^\Gamma(C, D \otimes \mathcal{T}(\varphi_0, \varphi_1)) & \xrightarrow{p_*} & KK^\Gamma(C, D \otimes A) \\ (\varphi_0)_* - (\varphi_1)_* \uparrow & & & & \downarrow (\varphi_0)_* - (\varphi_1)_* \\ KK^\Gamma(C, D \otimes SA) & \xleftarrow{p^*} & KK^{1, \Gamma}(C, D \otimes \mathcal{T}(\varphi_0, \varphi_1)) & \xleftarrow{h_*} & KK^\Gamma(C, D \otimes B). \end{array}$$

There is also an analogous exact sequence with reversed arrows for A, B and $\mathcal{T}(\varphi_0, \varphi_1)$ on the left hand side.

Proof. Although this follows from [18, 28] (at least in the non equivariant setting), we wish to outline here that this is a Puppe type exact sequence and holds automatically for every homotopy functor (see [11]). In particular, it holds for the equivariant KK^Γ -theory as we will use it.

Exactness at A : This is in a sense just tautological; for $x \in KK^\Gamma(C, D \otimes A)$, we have $(\varphi_0)_*(x) = (\varphi_1)_*(x)$ if and only if these elements are homotopic *i.e.* if x is in $p^*(KK^\Gamma(C, D \otimes \mathcal{T}(\varphi_0, \varphi_1)))$.

Exactness at \mathcal{T} : The mapping cone of p is $\mathcal{T}(\varphi_0 \circ e_0, \varphi_1 \circ e_0)$ where $e_0 : C_0([0, 1], A) \rightarrow A$ is the evaluation at 0. As e_0 is homotopic to the evaluation at 1, which is the 0 map, this torus algebra is homotopy equivalent to $\mathcal{T}(\varphi_0 \circ e_1, \varphi_1 \circ e_1) = SB \oplus C_0([0, 1], A)$ whence to SB . More precisely, the inclusion of the kernel SB of p in the mapping cone of p is a homotopy equivalence. Use then the Puppe sequence which holds in full generality.

Exactness at B : The mapping cone of p is $\mathcal{T}(\tilde{\varphi}_0, \tilde{\varphi}_1)$ where $\tilde{\varphi}_i : SA \rightarrow C_0([0, 1], B)$ are $f \mapsto \varphi_i \circ f$. These maps are again homotopic to 0, whence $\mathcal{T}(\tilde{\varphi}_0, \tilde{\varphi}_1)$ is homotopy equivalent to SA . \square

Remarks 1.3. (1) Let $\varphi_0, \varphi_1 : A \rightarrow B$ be morphisms and $u_0, u_1 \in B$ unitaries. Then we have a natural Morita equivalence between the torus algebras $\mathcal{T}(\varphi_0, \varphi_1)$ and $\mathcal{T}(\text{Ad}_{u_0} \circ \varphi_0, \text{Ad}_{u_1} \circ \varphi_1)$. Indeed the space $E = \{(a, f) \in A \times B[0, 1]; f(0) = u_0 \varphi_0(a); f(1) = u_1 \varphi_1(a)\}$ is the desired Morita equivalence bimodule.

If $[E]$ denotes the associated element in $KK(\mathcal{T}(\text{Ad}_{u_0} \circ \varphi_0, \text{Ad}_{u_1} \circ \varphi_1), \mathcal{T}(\varphi_0, \varphi_1))$, we have

$$p_*[E] = [p'_*], \quad (h')^*[E] = [h]$$

where $p : \mathcal{T}(\varphi_0, \varphi_1) \rightarrow A$ and $p' : \mathcal{T}(\text{Ad}_{u_0} \circ \varphi_0, \text{Ad}_{u_1} \circ \varphi_1) \rightarrow A$ denote the evaluation morphisms $(a, f) \mapsto a$, and $h : SB \rightarrow \mathcal{T}(\varphi_0, \varphi_1)$ and $h' : SB \rightarrow \mathcal{T}(\text{Ad}_{u_0} \circ \varphi_0, \text{Ad}_{u_1} \circ \varphi_1)$ denote the morphisms $f \mapsto (0, f)$.

(2) Let A_0 and A_1 be C^* -algebras with $\varphi_0 : A_0 \rightarrow B$ and $\varphi_1 : A_1 \rightarrow B$ be two morphisms. Recall that the associated double cylinder is defined by $Z(\varphi_0, \varphi_1) = \{(a_0, a_1, f) \in A_0 \times A_1 \times B[0, 1]; f(0) = \varphi_0(a_0) \text{ and } f(1) = \varphi_1(a_1)\}$. Given unitaries $u_0, u_1 \in B$, there is a canonical Morita equivalence $Z(\varphi_0, \varphi_1)$ and $Z(\text{Ad}_{u_0} \circ \varphi_0, \text{Ad}_{u_1} \circ \varphi_1)$. In fact, we may put $A = A_0 \times A_1$ and $\psi : A \rightarrow B$, $\psi_i(a_0, a_1) = \varphi_i(a_i), i = 0, 1$; then $Z(\varphi_0, \varphi_1)$ identifies with $\mathcal{T}(\psi_0, \psi_1)$.

1.3. KK -theory with real coefficients. It is natural to try to define the KK -theory with real coefficients as $KK(A, B \otimes M)$, where M is a II_1 -factor. On the other hand, it is not clear which M we should choose.

We give here a natural construction of $KK_{\mathbb{R}}$ taking into account *all possible* such M .

Take a separable Hilbert space H and consider the set $\mathcal{F}_{\text{II}_1}(H)$ of all II_1 -factors acting on H . Every factor $M \in \mathcal{F}_{\text{II}_1}(H)$ is endowed with its unique tracial state. Recall the following properties:

- (1) Every finite von Neumann algebra with trace can be embedded in a trace preserving way into a II_1 factor - just take for instance a free product with a II_1 -factor ([32], [23, Theorem 4.4], [12]).
- (2) For the same reason, a finite number of II_1 factors can be simultaneously embedded in a trace preserving way into a II_1 factor - just take their (tracial) free product.
- (3) Given two unital morphisms $\varphi, \psi: M_1 \rightarrow M_2$, there exists a II_1 -factor M_3 , namely (a II_1 -factor containing) the corresponding von Neumann HNN extension (see [14, §3]) and a unital embedding $\chi: M_2 \rightarrow M_3$ such that $\chi \circ \varphi$ and $\chi \circ \psi$ differ by an inner automorphism and thus define the same element of $KK(M_1, M_3)$.

Definition 1.4. For A and B separable C^* -algebras the diagram $\mathcal{F}_{\text{II}_1}(H) \ni M \mapsto KK(A, B \otimes M)$ with values groups has a well defined limit that we take as a definition of:

$$KK_{\mathbb{R}}(A, B) := \lim_{M \in \mathcal{F}_{\text{II}_1}(H)} KK(A, B \otimes M). \quad (1.1)$$

Construction 1.5. To define this limit it is convenient to build a direct system:

- for every M from $\mathcal{F}_{\text{II}_1}(H)$ define the group $\underline{KK}(A, B)[M]$ to be the quotient of $KK(A, B \otimes M)$ with respect to the subgroup of all elements $z \in KK(A, B \otimes M)$ which become zero under some unital embedding $\varphi: M \rightarrow N$. It is a well defined equivalence relation by the property (3) above.
- Every unital embedding $\varphi: M_1 \hookrightarrow M_2$ of II_1 -factors induces a group homomorphism $\varphi_*: \underline{KK}(A, B)[M_1] \rightarrow \underline{KK}(A, B)[M_2]$ sending the equivalence class of an element $[x] \in \underline{KK}(A, B \otimes M_1)$ to the equivalence class of $\varphi_*[x] \in \underline{KK}(A, B \otimes M_2)$. This map is well defined by property (2) and does not depend on φ by property (3).

We define a relation of partial pre-order in $\mathcal{F}_{\text{II}_1}(H)$ saying that $M_1 < M_2$ if there is a tracial embedding $M_1 \hookrightarrow M_2$. The partially pre-ordered set $(\mathcal{F}_{\text{II}_1}(H), <)$ is directed - by property (1) and the map $M \mapsto \underline{KK}(A, B)[M]$ is a direct system. It follows immediately that the unique limit L of $M \mapsto \underline{KK}(A, B)[M]$ is a limit for the diagram $M \mapsto KK(A, B \otimes M)$.

Remarks 1.6.

- (1) When A and B are in the bootstrap class, then $KK(A, B \otimes M)$ does not depend on the II_1 -factor M and is isomorphic to $\text{Hom}(K(A), K(B) \otimes \mathbb{R})$. In other words, for any II_1 -factor M the map $KK(A, B \otimes M) \rightarrow KK_{\mathbb{R}}(A, B)$ is an isomorphism.
- (2) If A and B are not in the bootstrap category, then there is no best choice for M . Indeed, the hyperfinite II_1 -factor is of course the smallest element of $(\mathcal{F}_{\text{II}_1}(H), <)$. By [22], there is no biggest element in $\mathcal{F}_{\text{II}_1}(H)$, and therefore there is no natural way to choose M and define $KK_{\mathbb{R}}(A, B) = KK(A, B \otimes M)$.

Note also that if M is a *big* II_1 -factor acting on a non separable Hilbert space, then for every separable C^* -algebra A , and $x \in KK(A, B \otimes M)$, there exists (by [29, Remark 3.2]) a separable subalgebra $N_1 \subset B \otimes M$ and $x_1 \in KK(A, N_1)$ whose image is x . We may then construct separable subalgebras $B_1 \subset B$ and $D_1 \subset M$ and hence a

(weakly) separable II_1 -subfactor $M_1 \subset M$ such that $N_1 \subset B_1 \otimes D_1 \subset B_1 \otimes M_1 \subset B \otimes M$. Let then \tilde{x}_1 denote the image of $x_1 \in KK_{\mathbb{R}}(A, B)$. If N_2, x_2 is another such choice, we may find a separable $N_3 \subset B \otimes M$ where x_1 and x_2 coincide and therefore a separable subfactor $M_3 \subset M$ such that $N_3 \subset B \otimes M_3$. It follows that $\tilde{x}_1 = \tilde{x}_2$. We thus obtain a natural morphism $KK(A, B \otimes M) \rightarrow KK_{\mathbb{R}}(A, B)$.

- (3) If we wish to avoid the use of non separable algebras and the method of [29] to treat the corresponding KK -products, we may also replace this category of II_1 -factors by the category of separable C^* -algebras with tracial states, the morphisms in this category being the trace preserving $*$ -homomorphisms. Define then $KK_{\mathbb{R}}(A, B)$ as the limit of $KK(A, B \otimes D)$ when D runs over all tracial separable C^* -algebras (acting on a given Hilbert space). We may form an inductive system $\underline{KK}(A, B)[D, \tau]$ as the quotient of $KK(A, B \otimes D)$ by the subgroup of x such that there exists a C^* -algebra D_1 with a tracial state τ_1 a trace preserving morphism $j : D \rightarrow D_1$ such that $j_*(x) = 0$.

It follows from [29] that the two definitions coincide.

- (4) The method used to show the existence of the limit in construction 1.5 can be summarized as follows: assume \mathcal{C} is a (small) category and $\mathcal{F} : \mathcal{C} \rightarrow \text{Grp}$ is a functor such that the following properties are satisfied:
- for every $A, B \in \text{Ob}_{\mathcal{C}}$ there exists some $C \in \text{Ob}_{\mathcal{C}}$ and arrows $A \rightarrow C$ and $B \rightarrow C$.
 - For every couple of arrows $f, g : A \rightarrow B$ there exists some $C \in \text{Ob}_{\mathcal{C}}$ and an arrow $h : B \rightarrow C$ with $\mathcal{F}(h \circ f) = \mathcal{F}(h \circ g)$.

Then we have shown that there is a natural transformation $\mu : \mathcal{F} \rightarrow \underline{\mathcal{F}}$ such that $\underline{\mathcal{F}}$ is a directed system and \mathcal{F} has a unique limit which is the unique limit of $\underline{\mathcal{F}}$.

1.4. Traces and KK -theory with real coefficients. A trace on a C^* -algebra D can be thought of as a generalized morphism to an abstract II_1 -factor. It therefore gives rise to a natural element of $KK_{\mathbb{R}}(D, \mathbb{C})$.

Definition 1.7. If D is a separable unital C^* -algebra and τ is a trace on D , then there is a II_1 -factor M and a finitely generated projective module E on M with a trace preserving morphism $f_{\tau} : D \rightarrow \mathcal{L}(E)$ (with von Neumann dimension $\dim_M(E) = \tau(1)$); in this way, we define an element $[\tau] \in KK_{\mathbb{R}}(D, \mathbb{C})$; using the morphism $f_{\tau} : D \rightarrow \mathcal{L}(E)$ we also obtain for every pair (A, B) of C^* -algebras a morphism

$$\tau_* : KK(A, B \otimes D) \rightarrow KK_{\mathbb{R}}(A, B).$$

In particular, we have $[\tau] = \tau_*(1_D) \in KK_{\mathbb{R}}(D, \mathbb{C})$.

1.5. The Kasparov product. Let A, B, D be C^* -algebras.

If $x \in KK_{\mathbb{R}}(A, D)$ and $y \in KK_{\mathbb{R}}(D, B)$, there exist II_1 -factors M and N such that x is the image of $x_0 \in KK(A, D \otimes M)$ and y is the image of $y_0 \in KK(D, B \otimes N)$. We then may form the KK -product and obtain an element $x_0 \otimes_D y_0 \in KK(A, B \otimes M \otimes N)$. We may then map the minimal tensor product $M \otimes N$ into the corresponding von Neumann tensor product and pass to the limit $KK_{\mathbb{R}}(A, B)$. We obtain an element $x \otimes_D y \in KK_{\mathbb{R}}(A, B)$ which only depends on x and y and not on (M, x_0) and (N, y_0) .

This Kasparov product immediately extends to a product

$$\otimes_D : KK_{\mathbb{R}}(A_1, B_1 \otimes D) \times KK_{\mathbb{R}}(D \otimes A_2, B_2) \rightarrow KK_{\mathbb{R}}(A_1 \otimes A_2, B_1 \otimes B_2)$$

which has all the usual properties of the Kasparov product (bilinearity, functoriality, associativity...).

Remark 1.8. Let τ be a trace on D ; by naturality of the Kasparov product, the map $\tau_* : KK(A, B \otimes D) \rightarrow KK_{\mathbb{R}}(A, B)$ is given by the product with $[\tau] \in KK_{\mathbb{R}}(D, \mathbb{C})$.

Remark 1.9. For every II_1 -factor M , the group $KK(\mathbb{C}, M)$ is canonically isomorphic with \mathbb{R} . It follows that the ring $KK_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ is naturally isomorphic with \mathbb{R} . We deduce that $KK_{\mathbb{R}}(A, B)$ is naturally a real vector space - and that the $KK_{\mathbb{R}}$ -product is \mathbb{R} -bilinear. If τ is a trace on a C^* -algebra D and $s \in \mathbb{R}_+$, then $[s\tau] = s[\tau] \in KK_{\mathbb{R}}(D, \mathbb{C})$.

1.6. KK -theory with \mathbb{R}/\mathbb{Z} coefficients. For every II_1 -factor M denote by $i_M : \mathbb{C} \rightarrow M$ the unital inclusion. For the \mathbb{R}/\mathbb{Z} -coefficients we can similarly define:

Definition 1.10. Let A, B be C^* -algebras we put:

$$KK_{\bullet, \mathbb{R}/\mathbb{Z}}(A, B) := \lim_{M \in \mathcal{F}_{\text{II}_1}(H)} KK_{\bullet+1}(A, B \otimes C_{i_M}). \quad (1.2)$$

As above, this is a limit along the partially ordered directed set $(\mathcal{F}_{\text{II}_1}(H), <)$.

The mapping cone exact sequence gives rise to a six term Bockstein change of coefficients exact sequence

$$\dots \rightarrow KK(A, B) \rightarrow KK_{\mathbb{R}}(A, B) \rightarrow KK_{\mathbb{R}/\mathbb{Z}}(A, B) \rightarrow \dots$$

Remark 1.11. Let A, B be C^* -algebras and M be a II_1 -factor. Denote by $i_M^B : B \rightarrow B \otimes M$ the map $x \mapsto x \otimes 1_M$. We have of course $C_{i_M^B} = B \otimes C_{i_M}$ and $Z(i_M^B, i_M^B) = B \otimes Z(i_M, i_M)$. Using the split exact sequence

$$0 \rightarrow C_{i_M^B} \rightarrow Z(i_M^B, i_M^B) \rightarrow B \rightarrow 0,$$

we find $KK(A, B \otimes Z(i_M, i_M)) = KK(A, B) \oplus KK(A, B \otimes C_{i_M})$. We thus get a morphism $\vartheta : KK(A, B \otimes Z(i_M, i_M)) \rightarrow KK_{\mathbb{R}/\mathbb{Z}}^1(A, B)$. Composing with the connecting map ∂ of the Bockstein change of coefficients exact sequence we obtain a map $\partial \circ \vartheta = (p_0)_* - (p_1)_* : KK(A, B \otimes Z(i_M, i_M)) \rightarrow KK(A, B)$, where $p_i : B \otimes Z(i_M, i_M) = Z(i_M^B, i_M^B) \rightarrow B$ are the maps $(b_0, b_1, f) \mapsto b_i$.

1.7. KK^Γ with coefficients. The above realizations of $KK_{\mathbb{R}}$ and $KK_{\mathbb{R}/\mathbb{Z}}$ can be extended to the equivariant setting.

Definition 1.12. Let Γ be a discrete group. We define $KK_{\mathbb{R}}^\Gamma(A, B)$ and $KK_{\mathbb{R}/\mathbb{Z}}^\Gamma(A, B)$ to be respectively the limit of $KK^\Gamma(A, B \otimes M)$ and $KK^\Gamma(A, B \otimes C_{i_M})$ where M runs over the set $\mathcal{F}_{\text{II}_1}(H)$ of II_1 -factors on the separable Hilbert space H - with trivial Γ action, $i_M : \mathbb{C} \rightarrow M$ denotes the unital inclusion and C_{i_M} its mapping cone.

This is again a direct limit along the partially preordered directed set $(\mathcal{F}_{\text{II}_1}(H), <)$.

Recall from [19] that Kasparov descent morphism is a natural morphism $j_\Gamma : KK^\Gamma(A, B) \rightarrow KK(A \rtimes \Gamma, B \rtimes \Gamma)$. If A, B are Γ -algebras and M a II_1 -factor (with trivial action of Γ), composing the descent morphism together with the natural morphism $(B \otimes M) \rtimes \Gamma \rightarrow (B \rtimes \Gamma) \otimes M$ we obtain a morphism

$$KK^\Gamma(A, B \otimes M) \rightarrow KK(A \rtimes \Gamma, (B \otimes M) \rtimes \Gamma) \rightarrow KK(A \rtimes \Gamma, (B \rtimes \Gamma) \otimes M).$$

This map passes to the inductive limit and defines a descent morphism with real coefficients $j_{\mathbb{R}}^\Gamma : KK_{\mathbb{R}}^\Gamma(A, B) \rightarrow KK_{\mathbb{R}}(A \rtimes \Gamma, B \rtimes \Gamma)$.

In the same way, taking coefficients in mapping cone algebras of inclusions $i_M : \mathbb{C} \rightarrow M$ and then inductive limit over $M \in \mathcal{F}_{\text{II}_1}(H)$, we obtain a descent morphism with \mathbb{R}/\mathbb{Z} coefficients $j_{\mathbb{R}/\mathbb{Z}}^\Gamma : KK_{\mathbb{R}/\mathbb{Z}}^\Gamma(A, B) \rightarrow KK_{\mathbb{R}/\mathbb{Z}}(A \rtimes \Gamma, B \rtimes \Gamma)$.

The Kasparov product

$$\otimes_D : KK_{\mathbb{R}}^{\Gamma}(A_1, B_1 \otimes D) \times KK_{\mathbb{R}}^{\Gamma}(D \otimes A_2, B_2) \rightarrow KK_{\mathbb{R}}^{\Gamma}(A_1 \otimes A_2, B_1 \otimes B_2)$$

is defined exactly as in Section 1.5 with all the functorial properties with respect to algebras.

Furthermore a morphism $\Gamma_1 \rightarrow \Gamma_2$ induces an obvious morphism $KK_{\mathbb{R}}^{\Gamma_2}(A, B) \rightarrow KK_{\mathbb{R}}^{\Gamma_1}(A, B)$.

Also, the mapping cone exact sequence in equivariant KK -theory gives rise to a six term Bockstein change of coefficients exact sequence

$$\dots \rightarrow KK^{\Gamma}(A, B) \rightarrow KK_{\mathbb{R}}^{\Gamma}(A, B) \rightarrow KK_{\mathbb{R}/\mathbb{Z}}^{\Gamma}(A, B) \rightarrow \dots$$

Remark 1.13. In this paper, we use the equivariant $KK_{\mathbb{R}}$ - and $KK_{\mathbb{R}/\mathbb{Z}}$ -theory with respect to a discrete group Γ . Of course, the same constructions can be performed equivariantly with respect to a locally compact group or a Hopf algebra (cf. [5]) or a groupoid (cf. [20]).

1.8. Some remarks on Γ -algebras. Let Γ be a discrete group. We end these comments with some easy observations on Γ -algebras.

Trivial action and equivariant Kasparov groups.

Lemma 1.14. *Let Γ be a discrete group, A a Γ -algebra and let B be a C^* -algebra endowed with the trivial action of Γ . Then there is a canonical isomorphism $KK^{\Gamma}(A, B) \cong KK(A \rtimes \Gamma, B)$.*

Proof. Note first that by [28, Lemma 1.11] one can take the action of A (resp. $A \rtimes \Gamma$) to be non-degenerate in the cycles defining $KK^{\Gamma}(A, B)$ (resp. $KK(A \rtimes \Gamma, B)$). With this in mind the rest of the proof is straightforward because Γ acts by unitary multipliers on $A \rtimes \Gamma$. \square

Remark 1.15. Note that the identification in Lemma 1.14 is the composition

$$KK^{\Gamma}(A, B) \xrightarrow{j_{\Gamma}} KK(A \rtimes \Gamma, B \otimes C^*\Gamma) \xrightarrow{(\text{id} \otimes \varepsilon)^*} KK(A \rtimes \Gamma, B)$$

where $\varepsilon : C^*\Gamma \rightarrow \mathbb{C}$ is the trivial representation.

The isomorphism $KK^{\Gamma}(A, B) \rightarrow KK(A \rtimes \Gamma, B)$ can be expressed as the composition of Kasparov's descent morphism $j_{\Gamma} : KK^{\Gamma}(A, B) \rightarrow KK(A \rtimes \Gamma, B \rtimes \Gamma)$ with the morphism $B \rtimes \Gamma = B \otimes_{\max} C^*\Gamma \rightarrow B$ associated with the trivial representation.

Inner action and equivariant Morita equivalence.

Lemma 1.16. *Let Γ be a discrete group, A a C^* -algebra and $g \mapsto u_g$ a morphism of Γ to the group of unitary multipliers of A . Let β be the corresponding inner action of Γ on A ($\beta(a) = u_g a u_g^*$). The Γ -algebras A endowed with the trivial action and A endowed with the action α are Morita equivalent in an equivariant way.*

Proof. We denote by A_1 and A_{β} the algebra A when endowed with the trivial action and the action β , respectively. Let E be the Hilbert A_1 -module A endowed with the action of Γ given by $g.x = u_g x$. As $u_g(xy) = \beta_g(x)u_g y$, the action of A by left multiplication on E is equivariant: it follows that E is a Γ -equivariant Morita equivalence between A_{β} and A_1 . \square

1.9. Kasparov's descent and coproducts. Let A be a Γ -algebra. Denote by $\delta^A : A \rtimes \Gamma \longrightarrow (A \rtimes \Gamma) \otimes C^*\Gamma$ the coaction map. Recall that $\delta^A(au_g) = au_g \otimes u_g$ for $a \in A \subset A \rtimes \Gamma$ and $g \in \Gamma$ where u_g denotes both the element of $C^*\Gamma$ and of (the multiplier algebra of) $A \rtimes \Gamma$ corresponding to g .

Remark 1.17. Let B be a C^* -algebra endowed with a trivial Γ -action. Through the identification $KK^\Gamma(A, B) \cong KK(A \rtimes \Gamma, B)$, Kasparov's descent morphism is given by $x \mapsto (\delta_{max}^A)^*(x \otimes_{max} 1_{C^*\Gamma}) : KK(A \rtimes \Gamma, B) \rightarrow KK(A \rtimes \Gamma, B \rtimes \Gamma) = KK(A \rtimes \Gamma, B \otimes_{max} C^*\Gamma)$. Here, we denoted by $\delta_{max}^A : A \rtimes \Gamma \rightarrow (A \rtimes \Gamma) \otimes_{max} C^*\Gamma$ the morphism defined by the same formula as δ^A .

2. K-THEORETICALLY "FREE AND PROPER" ALGEBRAS AND ρ CLASS

2.1. KK-theory elements with real coefficients associated with a trace on $C^*\Gamma$.

As explained in lemma 1.14, for every Γ -algebra A and every C^* -algebra B , we may identify $KK(A \rtimes \Gamma, B)$ with $KK^\Gamma(A, B)$ where B is endowed with the trivial action of B . Replacing in this formula B by $B \otimes M$ where M is a II_1 -factor and taking the inductive limit, we find for a Γ -algebra A and a C^* -algebra B endowed with a trivial Γ action an identification $KK_{\mathbb{R}}^\Gamma(A, B) = KK_{\mathbb{R}}(A \rtimes \Gamma, B)$ - and in the same way, $KK_{\mathbb{R}/\mathbb{Z}}^\Gamma(A, B) = KK_{\mathbb{R}/\mathbb{Z}}(A \rtimes \Gamma, B)$.

A trace τ on $C^*\Gamma$ gives rise to an element $[\tau] \in KK_{\mathbb{R}}(C^*\Gamma, \mathbb{C})$ (Definition 1.7) and therefore an element $[\tau]^\Gamma \in KK_{\mathbb{R}}^\Gamma(\mathbb{C}, \mathbb{C})$.

2.2. K-theoretically "free and proper" algebras. We now consider a class of Γ -algebras on the K -theory of which "the trace acts as the unit element". Those are the *K-theoretically "free and proper" algebras*:

Definition 2.1. Let Γ be a discrete group. Denote by \mathbf{tr} its group tracial state. We say that A satisfies property (KFP) if $1_A^\Gamma \otimes [\mathbf{tr}]^\Gamma$ is equal to the unit element $1_{A, \mathbb{R}}^\Gamma$ of the ring $KK_{\mathbb{R}}^\Gamma(A, A)$.

We will see in the following section important examples of algebras satisfying this property. In particular, we will see that free and proper algebras satisfy this condition (Theorem 3.10). Note however that, if Γ has a γ element in the sense of [19, 30], K -theoretically proper algebras should be those for which $\gamma = 1$. It is not at all clear to us whether condition (KFP) implies $\gamma = 1$.

On the other hand, it is worth noting that property (KFP) implies a kind of freeness condition: property (KFP) cannot hold for the algebra \mathbb{C} whenever the group Γ has torsion. Indeed, let $\varepsilon : C^*\Gamma \rightarrow \mathbb{C}$ be the trivial representation. The algebra \mathbb{C} has property (KFP) if and only if the classes $[\mathbf{tr}]$ and $[\varepsilon]$ in $KK_{\mathbb{R}}^\Gamma(\mathbb{C}, \mathbb{C}) = KK_{\mathbb{R}}(C^*\Gamma, \mathbb{C})$ coincide. Assume that $g \in \Gamma$ has finite order $m \neq 1$. Then the element $p = 1/m(1 + \gamma + \dots + \gamma^{m-1}) \in \mathbb{C}\Gamma$ is a projection with $[p] \otimes [\mathbf{tr}] = \mathbf{tr}(p) = 1/m$ and $[p] \otimes [\varepsilon] = \varepsilon(p) = 1$.

Assume A satisfies property (KFP). This means that there is a II_1 -factor N with a morphism $\lambda : C^*\Gamma \longrightarrow N$ such that $\text{tr}_N \circ \lambda = \mathbf{tr}$ (where tr_N is the normalized trace of N) and such that the $(A, A \otimes N)$ -bimodule $A \otimes N$ where A acts on the left by $a \mapsto a \otimes 1$ endowed with the action $g \mapsto g \otimes \lambda(g)$ and $g \mapsto g \otimes 1$ define the same element in $KK^\Gamma(A, A \otimes N)$.

Let us make a few comments on this definition:

Properties 2.2. (1) If A satisfies property (KFP) and B is any Γ -algebra, then $A \otimes B$ satisfies property (KFP).

Indeed, if $1_A^\Gamma \otimes [\mathbf{tr}]^\Gamma = 1_{A, \mathbb{R}}^\Gamma$, then $1_A^\Gamma \otimes 1_B^\Gamma \otimes [\mathbf{tr}]^\Gamma = (1_A^\Gamma \otimes [\mathbf{tr}]^\Gamma) \otimes 1_B^\Gamma = 1_{A \otimes B, \mathbb{R}}^\Gamma$.

- (2) Let A, B be Γ -algebras. If A is KK^Γ -subequivalent to B and B satisfies property (KFP) then so does A .

The assumption means that there exist $x \in KK^\Gamma(A, B)$ and $y \in KK^\Gamma(B, A)$ satisfying $x \otimes_B y = 1_A^\Gamma$. Note that $y \otimes_{\mathbb{C}} [\mathbf{tr}]^\Gamma = [\mathbf{tr}]^\Gamma \otimes_{\mathbb{C}} y = (1_B^\Gamma \otimes [\mathbf{tr}]^\Gamma) \otimes_B y$. If B satisfies property (KFP), then $y \otimes_{\mathbb{C}} [\mathbf{tr}]^\Gamma$ is the image in $KK_{\mathbb{R}}^\Gamma(B, A)$ of y , whence $1_A^\Gamma \otimes [\mathbf{tr}]^\Gamma = x \otimes_B (y \otimes_{\mathbb{C}} [\mathbf{tr}]^\Gamma) = 1_{A, \mathbb{R}}^\Gamma$.

Remark 2.3. Let τ, τ' be tracial states on $C^*\Gamma$. Let $[\tau]^\Gamma$ and $[\tau']^\Gamma$ denote their classes in $KK_{\mathbb{R}}^\Gamma(\mathbb{C}, \mathbb{C})$. Their Kasparov product $[\tau]^\Gamma \otimes [\tau']^\Gamma$ is then the class of the trace $\tau \cdot \tau' = (\tau \otimes \tau') \circ \delta$. Note that $\tau \cdot \tau'(g) = \tau(g)\tau'(g)$ for any group element $g \in \Gamma$, and in particular $\tau \cdot \mathbf{tr} = \mathbf{tr}$. If A satisfies property (KFP), then $1_A^\Gamma \otimes [\tau]^\Gamma = 1_{A, \mathbb{R}}^\Gamma$ for any tracial state τ .

Remark 2.4. If A satisfies property (KFP), then the morphism $q_\Gamma : A \rtimes \Gamma \rightarrow A \rtimes_{red} \Gamma$ is $KK_{\mathbb{R}}$ invertible.

Indeed, given a von Neumann algebra N and a morphism $\lambda : C^*\Gamma \rightarrow N$ such that $\text{tr}_N \circ \lambda = \mathbf{tr}$ (where tr_N is the normalized trace of N), the morphism $\lambda : C^*\Gamma \rightarrow N$ factors through $C_{red}^*(\Gamma)$; it therefore defines a morphism $\widehat{\delta}_\lambda^A : A \rtimes_{red} \Gamma \rightarrow A \rtimes \Gamma \otimes N$. We obviously have $q_\Gamma^*[\widehat{\delta}_\lambda^A] = j_\Gamma(1_A^\Gamma \otimes [\mathbf{tr}]^\Gamma)$ and $(q_\Gamma)_*[\widehat{\delta}_\lambda^A] = j_{\Gamma, red}(1_A^\Gamma \otimes [\mathbf{tr}]^\Gamma)$.

We will use the quite obvious following lemma:

Lemma 2.5. *Let $\alpha : \Gamma \rightarrow U_n$ be a unitary representation. The group Γ acts on $C(U_n)$ by left translation via α . Let $u : U_n \rightarrow M_n(\mathbb{C})$ be the identity of U_n seen as an element of $C(U_n) \otimes M_n(\mathbb{C})$. For $g \in \Gamma$, we denote by $\lambda_g \in C(U_n) \rtimes \Gamma$ the corresponding element. For $g \in \Gamma$ we have $(\lambda_g \otimes \alpha_g)u = u(\lambda_g \otimes 1)$ in $(C(U_n) \rtimes \Gamma) \otimes M_n(\mathbb{C})$.*

Proof. Note that, for $f \in C(U_n)$ and $x \in U_n$ we have $(\lambda_g f \lambda_g^{-1})(x) = f(\alpha_g^{-1}x)$. Write $u = \sum a_j \otimes b_j$ where $a_j \in C(U_n)$ and $b_j \in M_n(\mathbb{C})$. By definition of u , we have $\sum_j a_j(x)b_j = x$ for all $x \in U_n$.

We have $((\lambda_g \otimes 1)u(\lambda_g^{-1} \otimes 1))(x) = \sum_j a_j(\alpha_g^{-1}x)b_j = \alpha_g^{-1}x$, and the result follows. \square

2.3. Gluing Kasparov bimodules. To construct ρ_α^A , we glue two Kasparov bimodules on a mapping cylinder in such a way that they form a Kasparov bimodule on a double cylinder. We discuss here this general construction.

Let A, B_0, B be Γ -algebras and $j : B_0 \rightarrow B$ an equivariant morphism. Denote by $Z_j = \{(b, f) \in B_0 \times B[0, 1]; f(0) = j(b)\}$ the mapping cylinder of j and $Z_{j,j} = \{(b_0, b_1, f) \in B_0 \times B_0 \times B[0, 1]; f(0) = j(b_0), f(1) = j(b_1)\}$ its mapping double cylinder. Let $\overline{\text{ev}}_0 : Z_j \rightarrow B_0$ and $\text{ev}_1 : Z_j \rightarrow B$ be the maps $\overline{\text{ev}}_0 : (b, f) \mapsto b$ and $\text{ev}_1 : (b, f) \mapsto f(1)$.

We have a split exact sequence $0 \rightarrow C_j \rightarrow Z_{j,j} \xrightarrow{\overline{\text{ev}}_1} B_0 \rightarrow 0$ where $C_j = \{(b, f) \in B_0 \times B[0, 1]; f(0) = j(b)\}$ is the mapping cone of j and $\overline{\text{ev}}_1(a, b, f) = b$ - the section $\tilde{j} : B_0 \rightarrow Z_{j,j}$ is given by $\tilde{j}(b) = (b, b, f)$ where $f(t) = j(b)$ for all $t \in [0, 1]$. We thus have a group homomorphism $\Theta : KK^\Gamma(A, Z_{j,j}) \rightarrow KK^\Gamma(A, C_j)$ which is a left inverse of the inclusion $C_j \rightarrow Z_{j,j}$ and vanishes on the image of \tilde{j}_* .

We will identify the double cylinder $Z_{j,j}$ with the algebra $\mathcal{Z}_j = \{(x, y) \in Z_j \times Z_j; \text{ev}_1(x) = \text{ev}_1(y)\}$ through the isomorphism $\Phi : Z_{j,j} \rightarrow \mathcal{Z}_j$ given by: $\Phi(a, b, f) = ((a, g), (b, h))$ with $g(t) = f(t/2)$, $h(t) = f(1 - t/2)$ (for $t \in [0, 1]$).

Given a Hilbert Z_j -module E , the Hilbert B_0 -module $E \otimes_{\overline{\text{ev}}_0} B_0$ and the Hilbert- B -module $E \otimes_{\text{ev}_1} B$ are quotients of E . We denote by $\overline{\text{ev}}_0 : E \rightarrow E \otimes_{\overline{\text{ev}}_0} B_0$ and $\text{ev}_1 : E \rightarrow E \otimes_{\text{ev}_1} B$ the quotient maps.

Proposition 2.6. *Let $(E, F), (E', F')$ be Γ -equivariant Kasparov (A, Z_j) -bimodules.*

- (1) *Let $w \in \mathcal{L}(E' \otimes_{\text{ev}_1} B, E \otimes_{\text{ev}_1} B)$ be a unitary equivalence between the Γ -equivariant Kasparov (A, B) -bimodules $(\text{ev}_1)_*(E', F')$ and $(\text{ev}_1)_*(E, F)$. Define*

$$E \diamond_w E' = \{(\xi, \zeta) \in E \times E'; \text{ev}_1(\xi) = w \text{ev}_1(\zeta)\}.$$

It is naturally an equivariant Hilbert- (A, Z_j) -bimodule. The map $(F \diamond F') : (\xi, \zeta) \mapsto (F\xi, F'\zeta)$ is an (odd) element in $\mathcal{L}(E \diamond_w E')$. The pair $(E \diamond_w E', F \diamond F')$ is an equivariant Kasparov (A, Z_j) -bimodule

- (2) *If $v, w \in \mathcal{L}(E' \otimes_{\text{ev}_1} B, E \otimes_{\text{ev}_1} B)$ are two unitary equivalences, homotopic among unitary equivalences, then the classes of $(E \diamond_v E', F \diamond F')$ and $(E \diamond_w E', F \diamond F')$ in $KK^\Gamma(A, Z_j)$ coincide.*
- (3) *The class $\Theta[(E \diamond_{\text{id}} E, F \diamond F)]$ in $KK^\Gamma(A, C_j)$ is 0.*
- (4) *Let (E'', F'') be another Γ -equivariant Kasparov (A, Z_j) -bimodule. Let $w \in \mathcal{L}(E' \otimes_{\text{ev}_1} B, E \otimes_{\text{ev}_1} B)$ and $w' \in \mathcal{L}(E'' \otimes_{\text{ev}_1} B, E' \otimes_{\text{ev}_1} B)$ be unitary equivalences between the Γ -equivariant Kasparov (A, B) -bimodules $(\text{ev}_1)_*(E', F')$ and $(\text{ev}_1)_*(E, F)$ and between $(\text{ev}_1)_*(E'', F'')$ and $(\text{ev}_1)_*(E', F')$ respectively. Then*

$$\Theta[(E \diamond_{ww'} E'', F \diamond F'')] = \Theta[(E \diamond_w E', F \diamond F')] + \Theta[(E' \diamond_{w'} E'', F \diamond F'')].$$

Proof. (1) comes from the fact that w is supposed to commute with A, Γ and $wF' = Fw$. Also, it is easily seen that $\mathcal{K}(E \diamond_w E') = \{(T, T') \in \mathcal{K}(E) \times \mathcal{K}(E'); w(T' \otimes_{\text{ev}_1} 1) = (T \otimes_{\text{ev}_1} 1)w\}$.

(2) is immediate.

- (3) Let $(E_t, F_t)_{t \in [0,1]}$ be a homotopy joining $(\tilde{\mathbf{j}} \circ \overline{\text{ev}}_0)_*(E, F)$ to (E, F) : write $(E_t, F_t) = (h_t)_*(E, F)$ where $h_t : Z_j \rightarrow Z_j$ is the map $(b, f) \mapsto (b, f_t)$ where $f_t(s) = f(st)$. Then $(E_t \diamond_{\text{id}} E_t, F_t \diamond F_t)$ is a homotopy joining $(\tilde{\mathbf{j}} \circ \overline{\text{ev}}_0)_*(E, F)$ to $(E \diamond_{\text{id}} E, F \diamond F)$.

- (4) The unitary equivalences $u = \begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}, v = \begin{pmatrix} 0 & \text{id} \\ ww' & 0 \end{pmatrix} \in \mathcal{L}((E' \oplus E'') \otimes_{\text{ev}_1} B, (E \oplus E') \otimes_{\text{ev}_1} B)$ are homotopic among unitary equivalences. By (2) the classes

$$[((E \oplus E') \diamond_u (E' \oplus E''), (F \oplus F') \diamond (F' \oplus F''))]$$

and

$$[((E \oplus E') \diamond_v (E' \oplus E''), (F \oplus F') \diamond (F' \oplus F''))]$$

in $KK^\Gamma(A, Z_{j,j})$ coincide. But clearly

$$[((E \oplus E') \diamond_u (E' \oplus E''), (F \oplus F') \diamond (F' \oplus F''))] = [(E \diamond_w E', F \diamond F')] + [(E' \diamond_{w'} E'', F \diamond F'')]$$

and

$$[((E \oplus E') \diamond_v (E' \oplus E''), (F \oplus F') \diamond (F' \oplus F''))] = [(E \diamond_{ww'} E'', F \diamond F'')] + [(E' \diamond_{\text{id}} E', F' \diamond F')].$$

The result follows from (3). \square

2.4. Construction of ρ_α^A . Let A be a Γ -algebra satisfying property (KFP), and α a finite dimensional unitary representation of Γ . Denote by $[\alpha]^\Gamma \in KK^\Gamma(\mathbb{C}, \mathbb{C})$ its KK^Γ -class.

By Remark 2.3, $1_{A, \mathbb{R}}^\Gamma \otimes [\alpha] = n1_{A, \mathbb{R}}^\Gamma$ where $1_{A, \mathbb{R}}^\Gamma$ denotes the unit element of the ring $KK_{\mathbb{R}}^\Gamma(A, A)$. Using the Bockstein change of coefficients exact sequence, it follows that there exists $z \in KK_{\mathbb{R}/\mathbb{Z}}^{1, \Gamma}(A, A)$ whose image in $KK^\Gamma(A, A)$ is $[\alpha] - n1_A^\Gamma$.

We now construct a class $\rho_\alpha^A \in KK_{\mathbb{R}/\mathbb{Z}}^{1, \Gamma}(A, A)$ whose image in $KK_{\mathbb{R}/\mathbb{Z}}^{1, \Gamma}(A, A)$ is $1_{A, \mathbb{R}}^\Gamma \otimes [\alpha]^\Gamma - n1_{A, \mathbb{R}}^\Gamma$.

The important fact in our construction is that the element ρ_α^A is *independent of all choices*.

The assumption (KFP) says that there exists a II_1 -factor N (with trivial action of Γ), with a tracial morphism $\lambda = \lambda_N : C^*\Gamma \longrightarrow N$, *i.e.* a morphism such that $\text{tr}_N \circ \lambda = \mathbf{tr}$ (where tr_N is the normalized trace of N) satisfying

$$1_A^\Gamma \otimes [i] = 1_A^\Gamma \otimes [P_\lambda]$$

where $i = i_N : \mathbb{C} \rightarrow N$ is the unital morphism and P_λ is the Γ -equivariant Hilbert \mathbb{C}, N -bimodule N with the natural Hilbert N -module structure, and where $g \in \Gamma$ acts by left multiplication by $\lambda(g)$. Under the isomorphism $KK^\Gamma(\mathbb{C}, N) = KK(C^*\Gamma, N)$ the class $[P_\lambda]$ corresponds to the class of the morphism λ .

There exists a II_1 -factor M containing N with a morphism $C(U_n) \rtimes \Gamma \rightarrow M$ extending the morphism $\lambda : C^*\Gamma \rightarrow N \subset M$. Indeed we can take as M (a II_1 -factor containing) the free product of N with $L^\infty(U_n) \rtimes \Gamma$ with amalgamation over the group von Neumann algebra of Γ .

Up to replacing N by M , it follows, thanks to Lemma 2.5, that we may assume that there exists a unitary $u \in N \otimes M_n(\mathbb{C})$ satisfying $(\lambda_g \otimes \alpha_g)u = u(\lambda_g \otimes 1) \in N \otimes M_n(\mathbb{C})$ for all $g \in \Gamma$.

Denote by V_α the equivariant (\mathbb{C}, \mathbb{C}) bimodule \mathbb{C}^n where Γ acts through the action α . Denote also by V_n the equivariant (\mathbb{C}, \mathbb{C}) bimodule \mathbb{C}^n where Γ acts trivially. We have a unitary equivalence $1_A \otimes u \in \mathcal{L}(A \otimes P_\lambda \otimes V_n, A \otimes P_\lambda \otimes V_\alpha)$ between the Γ -equivariant Kasparov $(A, A \otimes N)$ bimodules $(A \otimes P_\lambda \otimes V_n, 0)$ and $(A \otimes P_\lambda \otimes V_\alpha, 0)$.

Let $i : \mathbb{C} \rightarrow N$ be the unital inclusion. Denote by $\bar{p}_0 : A \otimes Z_i \rightarrow A$ and $p_1 : A \otimes Z_i \rightarrow A \otimes N$ the maps $\text{id}_A \otimes \bar{e}v_0$ and $\text{id}_A \otimes e v_1$, where $\bar{e}v_0 : Z_i \rightarrow \mathbb{C}$ is the map $(a, f) \mapsto a$ and $e v_1 : Z_i \rightarrow N$ is the map $(a, f) \mapsto f(1)$.

As $1_A^\Gamma \otimes [i] = 1_A^\Gamma \otimes [P_\lambda]$, there exists an equivariant Kasparov-bimodule $(E, F) \in \mathbf{E}^\Gamma(A, A \otimes Z_i)$ joining $(A, 0)$ and $(A \otimes P_\lambda, 0)$. More precisely the equivariant Kasparov bimodule (E, F) is such that:

- the induced equivariant bimodule $E \otimes_{\bar{p}_0} A$ is the (A, A) -bimodule A (endowed with the natural Γ -action - and $F_0 = 0$);
- the induced equivariant bimodule $E \otimes_{p_1} (A \otimes N)$ is the equivariant $(A, A \otimes N)$ -bimodule $A \otimes P_\lambda$.

We will say that a “*proof of the property (KFP)*” for A consists of $N, \lambda, (E, F)$ where:

- N is a II_1 -factor with normalized trace tr_N ;
- $\lambda : C^*\Gamma \longrightarrow N$, is a morphism such that $\text{tr}_N \circ \lambda = \mathbf{tr}$;
- $(E, F) \in \mathbf{E}^\Gamma(A, A \otimes Z_i)$ is a Kasparov bimodule joining $(A, 0)$ and $(A \otimes P_\lambda, 0)$.

Theorem 2.7. (1) *Let $(E, F) \in \mathbf{E}^\Gamma(A, A \otimes Z_i)$ be a Kasparov bimodule joining $(A, 0)$ and $(A \otimes P_\lambda, 0)$. The class in $KK^\Gamma(A, A \otimes Z_{i,i})$ of*

$$(E \otimes V_\alpha, F \otimes 1) \diamond_{\text{id}_A \otimes u} (E \otimes V_n, F \otimes 1)$$

does not depend on the unitary $u \in N \otimes M_n(\mathbb{C})$ satisfying $(\lambda_g \otimes \alpha_g)u = u(\lambda_g \otimes 1)$ for all $g \in \Gamma$.

(2) *The image of this class in $KK_{\mathbb{R}/\mathbb{Z}}^{1,\Gamma}(A, A)$ through the composition*

$$\vartheta : KK^\Gamma(A, A \otimes Z_{i,i}) \xrightarrow{\Theta} KK^\Gamma(A, A \otimes C_i) \longrightarrow KK_{\mathbb{R}/\mathbb{Z}}^{1,\Gamma}(A, A)$$

does not depend:

- (a) *on $(E, F) \in \mathbf{E}^\Gamma(A, A \otimes Z_i)$ such that $E \otimes_{A \otimes Z_i} A = A$ and $E \otimes_{A \otimes Z_i} A = A \otimes P_\lambda$;*

- (b) on the II_1 -factor N containing a tracial morphism of $\lambda_N : C^*\Gamma \rightarrow N$ and a unitary $u \in N \otimes M_n(\mathbb{C})$ satisfying
- (i) $1_A^\Gamma \otimes [i] = 1_A^\Gamma \otimes [\lambda_N]$
 - (ii) $(\lambda_g \otimes \alpha_g)u = u(\lambda_g \otimes 1) \in N \otimes M_n(\mathbb{C})$ for all $g \in \Gamma$.

Proof. (1) Choose a $u \in N \otimes M_n(\mathbb{C})$ such that $(\lambda_g \otimes \alpha_g)u = u(\lambda_g \otimes 1) \in N \otimes M_n(\mathbb{C})$ for all $g \in \Gamma$. Then an element $v \in N \otimes M_n(\mathbb{C})$ satisfies the same property if and only if it is of the form uw where w is a unitary of the commutant of $\lambda(\Gamma) \otimes 1$ in $N \otimes M_n(\mathbb{C})$. But the set of unitaries in the von Neumann algebra $(N \cap \lambda(G)') \otimes M_n(\mathbb{C})$ is connected.

- (2) (a) Let $(E', F') \in \mathbf{E}^\Gamma(A, A \otimes Z_i)$ be another Kasparov bimodule joining $(A, 0)$ and $(A \otimes P_\lambda, 0)$. Choose a $u \in N \otimes M_n(\mathbb{C})$ such that $(\lambda_g \otimes \alpha_g)u = u(\lambda_g \otimes 1) \in N \otimes M_n(\mathbb{C})$ for all $g \in \Gamma$ and put $\tilde{u} = \text{id}_A \otimes u$.

By Proposition 2.6(3) and 2.6(4), the image by Θ of $(E \otimes V_\alpha, F \otimes 1) \diamond_{\tilde{u}} (E \otimes V_n, F \otimes 1)$ is opposite to the one of $(E \otimes V_n, F \otimes 1) \diamond_{\tilde{u}^*} (E \otimes V_\alpha, F \otimes 1)$; moreover, by Proposition 2.6(4), the image by Θ of

$$[(E' \otimes V_\alpha, F' \otimes 1) \diamond_{\tilde{u}} (E' \otimes V_n, F' \otimes 1)] + [(E \otimes V_n, F \otimes 1) \diamond_{\tilde{u}^*} (E \otimes V_\alpha, F \otimes 1)]$$

coincides with that of

$$[(E' \otimes V_\alpha, F' \otimes 1) \diamond_{\text{id}} (E \otimes V_\alpha, F \otimes 1)] + [(E \otimes V_n, F \otimes 1) \diamond_{\text{id}} (E' \otimes V_n, F' \otimes 1)].$$

Using again 2.6(3) and 2.6(4), we may replace $[(E \otimes V_n, F \otimes 1) \diamond_{\text{id}} (E' \otimes V_n, F' \otimes 1)]$ by $-[(E' \otimes V_n, F' \otimes 1) \diamond_{\text{id}} (E \otimes V_n, F \otimes 1)]$. We end up with

$$\Theta([(E, F) \diamond_{\text{id}} (E', F')] \otimes_{\mathbb{C}} ([V_\alpha] - [V_n])).$$

Now, since $(\hat{p}_0)_*[(E, F) \diamond_{\text{id}} (E', F')] = (\hat{p}_0)_*[(E, F)] = 1_A = (\hat{p}_0)_*[(E', F')] = (\hat{p}_1)_*[(E, F) \diamond_{\text{id}} (E', F')]$, it follows that $\Theta([(E, F) \diamond_{\text{id}} (E', F')])$ is in the image of an element $x \in KK^\Gamma(A, A \otimes SN)$ via the inclusion $h : SN \rightarrow C_i$. As $([V_\alpha] - [V_n])$ is the 0 element in $KK_{\mathbb{R}}^\Gamma(A, A)$ it follows that the image of $x \otimes_{\mathbb{C}} ([V_\alpha] - [V_n])$ in $KK_{\mathbb{R}}^{1, \Gamma}(A, A)$ is the zero element.

- (b) Let $i_{M,N} : N \rightarrow M$ be a morphism of II_1 -factors. Denote by $i_{M,N} : Z_{i_N} \rightarrow Z_{i_M}$ and $i_{M,N} : Z_{i_N, i_N} \rightarrow Z_{i_M, i_M}$ the corresponding morphism, $v = (i_{M,N} \otimes \text{id})(u)$. Let $(E, F) \in \mathbf{E}^\Gamma(A, A \otimes Z_{i_N})$ such that $E \otimes_{A \otimes Z_{i_N}} A = A$ and $E \otimes_{A \otimes Z_{i_N}} A = A \otimes P_{\lambda_N}$. Put $(E_M, F_M) = (E, F) \otimes_{i_{M,N}} Z_{i_M}$. One checks immediately that the image by $i_{M,N} : Z_{i_N, i_N} \rightarrow Z_{i_M, i_M}$ of $[(E \otimes V_\alpha, F \otimes 1) \diamond_{\tilde{u}} (E \otimes V_n, F \otimes 1)]$ is

$$[(E_M \otimes V_\alpha, F_M \otimes 1) \diamond_{\tilde{v}} (E_M \otimes V_n, F_M \otimes 1)].$$

□

Definition 2.8. Let A a Γ -algebra satisfying property (KFP). We denote by $\rho_\alpha^A \in KK_{\mathbb{R}/\mathbb{Z}}^{1, \Gamma}(A, A)$ the image by the map $\vartheta : KK^\Gamma(A, A \otimes Z_{i,i}) \rightarrow KK_{\mathbb{R}/\mathbb{Z}}^{1, \Gamma}(A, A)$ of an element $[(E \otimes V_\alpha, F \otimes 1) \diamond_{\tilde{u}} (E \otimes V_n, F \otimes 1)] \in KK^\Gamma(A, A \otimes Z_{i,i})$ as in Theorem 2.7.

Proposition 2.9. The image of ρ_α^A under the connecting map of the Bockstein change of coefficients exact sequence $KK_{\mathbb{R}/\mathbb{Z}}^{1, \Gamma}(A, A) \rightarrow KK^\Gamma(A, A)$ is $[\alpha]^\Gamma - n1_A^\Gamma$.

Proof. Let N, u and (E, F) be a “proof of property (KFP) for A ”, as in Theorem 2.7. Then, by Remark 1.11, the image of $\rho_\alpha^A = \vartheta([(E \otimes V_\alpha, F \otimes 1) \diamond_{\tilde{u}} (E \otimes V_n, F \otimes 1)])$ by this connecting map is

$$(\bar{p}_0)_*[(E \otimes V_\alpha, F \otimes 1) \diamond_{\tilde{u}} (E \otimes V_n, F \otimes 1)] - (\bar{p}_1)_*[(E \otimes V_\alpha, F \otimes 1) \diamond_{\tilde{u}} (E \otimes V_n, F \otimes 1)].$$

But $(\bar{p}_0)_*[(E \otimes V_\alpha, F \otimes 1) \diamond_{\tilde{u}} (E \otimes V_n, F \otimes 1)] = (\bar{p}_0)_*[(E \otimes V_\alpha, F \otimes 1)] = 1_A^\Gamma \otimes [\alpha]^\Gamma$ and $(\bar{p}_1)_*[(E \otimes V_\alpha, F \otimes 1) \diamond_{\tilde{u}} (E \otimes V_n, F \otimes 1)] = (\bar{p}_0)_*[(E \otimes V_n, F \otimes 1)] = n \cdot 1_A^\Gamma$. \square

2.5. Properties of the construction ρ_α^A .

2.5.1. Dependence on α .

Proposition 2.10. *Let A be a Γ -algebra satisfying property (KFP). Let α_1, α_2 be finite dimensional unitary representations of Γ . We have*

- (1) $\rho_{\alpha_1 \oplus \alpha_2}^A = \rho_{\alpha_1}^A + \rho_{\alpha_2}^A$
- (2) $\rho_{\alpha_1 \otimes \alpha_2}^A = \dim \alpha_1 \cdot \rho_{\alpha_2}^A + \rho_{\alpha_1}^A \otimes_{\mathbb{C}} [\alpha_2] = \dim \alpha_2 \cdot \rho_{\alpha_1}^A + [\alpha_1] \otimes_{\mathbb{C}} \rho_{\alpha_2}^A$.

Proof. The first statement is obvious.

For the second one, put $n_i = \dim \alpha_i$. Let (E, F) be as in Theorem 2.7. Thanks to Proposition 2.6(4), we may write

$$\begin{aligned} (E \otimes V_{\alpha_1 \otimes \alpha_2}, F \otimes 1) \diamond (E \otimes V_{n_1 n_2}, F \otimes 1) &= \left((E \otimes V_{\alpha_1}, F \otimes 1) \diamond (E \otimes V_{n_1}, F \otimes 1) \right) \otimes V_{\alpha_2} \\ &\quad + \left((E \otimes V_{\alpha_2}, F \otimes 1) \diamond (E \otimes V_{n_2}, F \otimes 1) \right) \otimes V_{n_1} \end{aligned}$$

and the result follows. \square

2.5.2. Change of algebra. Kasparov product.

Proposition 2.11. *Let A, B be Γ -algebras satisfying property (KFP). Let $f : A \rightarrow B$ be an equivariant homomorphism. For every finite dimensional unitary representation α of Γ , we have $f_*(\rho_\alpha^A) = f^*(\rho_\alpha^B) \in KK_{\mathbb{R}/\mathbb{Z}}^{1, \Gamma}(A, B)$.*

This is a particular case of the following more general result.

Proposition 2.12. *Let A, B be Γ -algebras satisfying property (KFP) and $x \in KK^\Gamma(A, B)$. For every finite dimensional unitary representation α of Γ , we have $\rho_\alpha^A \otimes_A x = x \otimes_B \rho_\alpha^B \in KK_{\mathbb{R}/\mathbb{Z}}^{1, \Gamma}(A, B)$.*

Proof. Let $(E, F) \in \mathbf{E}^\Gamma(A, B)$ representing x .

Let $N, \lambda, (E_A, F_A)$ be a “proof of the property (KFP) for A ”. As $(E_A, F_A) \in \mathbf{E}^\Gamma(A, A \otimes Z_i)$ is a Kasparov bimodule joining A to $A \otimes P_\lambda$, the Kasparov product $(E_A, F_A) \otimes_A (E, F)$ can be represented by a Kasparov bimodule $(\tilde{E}, \tilde{F}) \in \mathbf{E}^\Gamma(A, B \otimes Z_i)$ such that $\bar{p}_0(\tilde{E}, \tilde{F}) = (E, F)$ and $p_1(\tilde{E}, \tilde{F}) = (E \otimes P_\lambda, F \otimes 1)$. It is then immediately checked (using for instance the connexion and positivity approach of [9]) that

$$[(\tilde{E} \otimes V_\alpha, \tilde{F} \otimes 1) \diamond_{1_E \otimes u} (\tilde{E} \otimes V_n, \tilde{F} \otimes 1)]$$

is a Kasparov product

$$[(E_A \otimes V_\alpha, F_A \otimes 1) \diamond_{1_E \otimes u} (E \otimes V_n, F \otimes 1)] \otimes_A [(E, F)] \in \mathbf{E}^\Gamma(A, A \otimes Z_{i,i}).$$

Therefore $\rho_\alpha^A \otimes_A x$ is the image of the class of

$$[(\tilde{E} \otimes V_\alpha, \tilde{F} \otimes 1) \diamond_{1_E \otimes u} (\tilde{E} \otimes V_n, \tilde{F} \otimes 1)]$$

in $KK_{\mathbb{R}/\mathbb{Z}}^{1, \Gamma}(A, B)$.

In the same way, let $M, \lambda_M, (E_B, F_B)$ be a “proof of the property (KFP) for B ”. We may of course assume (replacing M and N if necessary by a factor containing the free product amalgamated over the group von Neumann algebra of Γ) that $M = N$ and $\lambda_M = \lambda$.

In the same way as above, the Kasparov product $(E, F) \otimes_B (E_A, F_A)$ can be represented by a Kasparov bimodule $(\widehat{E}, \widehat{F}) \in \mathbf{E}^\Gamma(A, B \otimes Z_i)$ such that $\bar{p}_0(\widehat{E}, \widehat{F}) = (E, F)$ and $p_1(\widehat{E}, \widehat{F}) = (E \otimes P_\lambda, F \otimes 1)$. It follows that $x \otimes_B \rho_B^\alpha$ is the image of the class

$$[(\widehat{E} \otimes V_\alpha, \widehat{F} \otimes 1) \diamond_{1_E \otimes u} (\widehat{E} \otimes V_n, \widehat{F} \otimes 1)]$$

in $KK_{\mathbb{R}/\mathbb{Z}}^{1,\Gamma}(A, B)$.

To conclude, note that it follows from the proof of Theorem 2.7 (2a) that the image of

$$[(\check{E} \otimes V_\alpha, \check{F} \otimes 1) \diamond_{1_E \otimes u} (\check{E} \otimes V_n, \check{F} \otimes 1)]$$

in $KK_{\mathbb{R}/\mathbb{Z}}^{1,\Gamma}(A, B)$ does not depend on $(\check{E}, \check{F}) \in \mathbf{E}^\Gamma(A, B \otimes Z_i)$ joining (E, F) to $(E, F) \otimes P_\lambda$. \square

3. EXAMPLES

3.1. The Mishchenko bundle. As explained in the introduction, our starting example comes from the Mishchenko bundle associated with a Galois cover of a compact manifold.

Proposition 3.1. *Assume that the group Γ acts freely, properly and cocompactly on a smooth manifold \tilde{V} . Then $C_0(\tilde{V})$ satisfies property (KFP).*

Proof. In [1, Proposition 5.1], we noted that there is a II_1 -factor N with a morphism $\lambda : \Gamma \rightarrow N$ such that $\text{tr}_N \circ \lambda = \mathbf{tr}$ together with a continuous map ψ from \tilde{V} to the unitary group $U(N)$ of N such that $\psi(gx) = \lambda(g)\psi(x)$ for all $g \in G$ and $x \in \tilde{V}$. It gives rise to a unitary $w \in \mathcal{L}(C_0(\tilde{V}) \otimes N)$ intertwining the Γ -actions $\beta \otimes \text{id}_N$ and $\beta \otimes \lambda$ on $C_0(\tilde{V}) \otimes N$ where β is the action of Γ on $C_0(\tilde{V})$ by translation. Now the Hilbert module $C_0(\tilde{V}) \otimes N$ endowed with the left action of $C_0(\tilde{V})$ equal to the right one and the Γ -action $\beta \otimes \text{id}_N$ (and $\beta \otimes \lambda$) represent respectively the images $1_{C_0(\tilde{V})}^\Gamma$ and $1_{C_0(\tilde{V})}^\Gamma \otimes \mathbf{tr}$ in $KK_{\mathbb{R}}^\Gamma(C_0(\tilde{V}), C_0(\tilde{V}))$.

These equivariant bimodules being unitarily equivalent they represent the same class in $KK_{\mathbb{R}}^\Gamma(C_0(\tilde{V}), C_0(\tilde{V}))$. \square

Thus, for every finite dimensional unitary representation $\alpha : \Gamma \rightarrow U_n$, we get an element $\rho_\alpha^{C_0(\tilde{V})} \in KK_{\mathbb{R}/\mathbb{Z}}^{1,\Gamma}(C_0(\tilde{V}), C_0(\tilde{V}))$.

We now explain the relation of this element with the element $[\alpha]_{\text{APS}} \in K_{\mathbb{R}/\mathbb{Z}}^1(C(V))$ constructed by Atiyah Patodi and Singer [3, 4].

Let Y be a locally compact space on which Γ acts and A and B be $C_0(Y) - \Gamma$ -algebras. Denote by $\mathcal{R}KK^\Gamma(Y; A, B)$ the Kasparov equivariant KK -theory group defined in [19, §2]. In the language of KK -theory equivariant by groupoids introduced by Le Gall ([20]), $\mathcal{R}KK^\Gamma(Y; A, B) = KK^{\tilde{V} \rtimes \Gamma}(A, B)$.

If Γ acts freely and properly on Y there is a descent isomorphism (see [19])

$$\lambda^\Gamma : \mathcal{R}KK^\Gamma(Y; A, B) \longrightarrow \mathcal{R}(Y/\Gamma; A^\Gamma, B^\Gamma)$$

where A^Γ and B^Γ are the algebras of *invariant elements*. In the language of [20], this is the isomorphism corresponding to the Morita equivalent groupoids $Y \rtimes \Gamma$ and Y/Γ .

Let X be a compact space, A a C^* -algebra and B be a $C(X)$ -algebra. Then we have a canonical isomorphism $\mathcal{R}KK(X; A \otimes C(X), B) = KK(A, B)$.

Let Y be a free and proper and cocompact Γ -space. Put $X = Y/\Gamma$. Let A be a C^* -algebra and let B be a $C_0(Y) - \Gamma$ -algebra. We have a sequence of isomorphisms

$$KK(A, B^\Gamma) \simeq \mathcal{R}KK(X; A \otimes C(X), B^\Gamma) \simeq \mathcal{R}KK^\Gamma(Y; A \otimes C_0(Y), B).$$

Using the forgetful map $\mathcal{R}KK^\Gamma(Y; A \otimes C_0(Y), B) \rightarrow KK^\Gamma(A \otimes C_0(Y), B)$ we obtain a morphism $U_{A,B}^{Y,\Gamma} : KK(A, B^\Gamma) \rightarrow KK^\Gamma(A \otimes C_0(Y), B)$.

Using the Morita equivalence of $C_0(Y) \rtimes \Gamma$ with $C(X)$ and of $B \rtimes \Gamma$ with B^Γ we also have a sequence of maps

$$KK^\Gamma(A \otimes C_0(Y), B) \xrightarrow{j_\Gamma} KK(A \otimes C_0(Y) \rtimes \Gamma, B \rtimes \Gamma) \simeq KK(A \otimes C(X), B^\Gamma) \xrightarrow{i_X^*} KK(A, B^\Gamma)$$

where $i_X : \mathbb{C} \rightarrow C(X)$ is the unital inclusion. We thus obtain a morphism:

$$V_{A,B}^{Y,\Gamma} : KK^\Gamma(A \otimes C_0(Y), B) \rightarrow KK(A, B^\Gamma).$$

Lemma 3.2. *The composition $V_{A,B}^{Y,\Gamma} \circ U_{A,B}^{Y,\Gamma}$ is the identity of $KK(A, B^\Gamma)$.*

Proof. Since $X = Y/\Gamma$ is compact, B^Γ is the set of multipliers b of B which are Γ -invariant and satisfy $fb \in B$ for all $f \in C_0(Y)$. If E is a Γ -equivariant Hilbert B module, we may in the same way define the Hilbert B^Γ module E^Γ , for instance by considering E as a subspace of the Γ - $C_0(Y)$ -algebra $\mathcal{K}(E \oplus B)$ and then $E^\Gamma \subset \mathcal{K}(E \oplus B)^\Gamma$.

We start with a pair (E, F) representing an element in $\mathcal{R}KK^\Gamma(Y; A \otimes C_0(Y), B)$; by properness of the action (or, equivalently using the isomorphism $\mathcal{R}KK(X; A \otimes C(X), B^\Gamma) \simeq \mathcal{R}KK^\Gamma(Y; A \otimes C_0(Y), B)$), we may assume F is Γ -invariant. It then defines an element $F_0 \in \mathcal{L}(E^\Gamma)$. The elements (E^Γ, F_0) and (E, F) correspond to each other via the isomorphism $\mathcal{R}KK(X; A \otimes C(X), B^\Gamma) \simeq \mathcal{R}KK^\Gamma(Y; A \otimes C_0(Y), B)$.

The element corresponding to (E^Γ, F_0) via the isomorphism $\mathcal{R}KK(X; A \otimes C(X), B^\Gamma) \simeq KK(A, B^\Gamma)$ is of course (E^Γ, F_0) where we just do not consider the left $C(X)$ action !

The element corresponding to (E, F) under the forgetful morphism $\mathcal{R}KK^\Gamma(Y; A \otimes C_0(Y), B) \rightarrow KK^\Gamma(A \otimes C_0(Y), B)$ is of course (E, F) ; its image by j_Γ is $(E \rtimes \Gamma, F \otimes 1)$ where $E \rtimes \Gamma = E \otimes_B (B \rtimes \Gamma)$. The Hilbert $(C(X) \otimes A, B^\Gamma)$ bimodule corresponding to it under the Morita equivalences, is then seen to be E^Γ - and the Lemma follows. \square

According to Remark 1.13, we define $\mathcal{R}KK_{\mathbb{R}}^\Gamma(V; A, B)$ and $\mathcal{R}KK_{\mathbb{R}/\mathbb{Z}}^\Gamma(V; A, B)$ using inductive limits over II_1 -factors.

The APS element in $K_{\mathbb{R}/\mathbb{Z}}^1(V)$ was described in [1] as an element $[\alpha]_{\text{APS}} \in K_0(Z_{i,i} \otimes C(V))$.

Proposition 3.3. *The element $\rho_\alpha^{C_0(\tilde{V})}$ is the image of $[\alpha]_{\text{APS}}$ under the composition of the isomorphisms*

$$K_{1,\mathbb{R}/\mathbb{Z}}(C(V)) \simeq \mathcal{R}KK_{\mathbb{R}/\mathbb{Z}}^1(V; C(V), C(V)) \simeq \mathcal{R}KK_{\mathbb{R}/\mathbb{Z}}^{1,\Gamma}(\tilde{V}; C_0(\tilde{V}), C_0(\tilde{V}) \otimes Z_{i,i})$$

with the forgetful map

$$\mathcal{R}KK_{\mathbb{R}/\mathbb{Z}}^{1,\Gamma}(\tilde{V}; C_0(\tilde{V}), C_0(\tilde{V})) \rightarrow KK_{\mathbb{R}/\mathbb{Z}}^{1,\Gamma}(C_0(\tilde{V}), C_0(\tilde{V})).$$

Proof. In [1, Prop. 5.1 and Prop. 5.2], we represented the class $[\alpha]_{\text{APS}}$ as the image under the morphism $K_0(C(V) \otimes Z_{i,i}) \rightarrow K_{1,\mathbb{R}/\mathbb{Z}}(C(V))$ of the $C(V) \otimes Z_{i,i}$ module $\mathcal{E} = \{f \in C([0, 1], E^+ \otimes N); f(0) \in E^+ \otimes 1, w_v f(1) \in E^- \otimes 1\}$ where N is a II_1 -factor with tracial inclusion $\lambda : \Gamma \rightarrow N$, $i : \mathbb{C} \rightarrow N$ is the unital inclusion, E^+ is the flat vector bundle associated with α and E^- is the trivial vector bundle with rank equal to the dimension of α . The unitary w_v is given by $w_v := (\varphi \otimes 1_{E^-})^{-1} \circ v^{-1} \circ (\varphi \otimes 1_{E^+})$ where $\varphi : C(V) \otimes N \rightarrow \mathcal{E}$ is an isomorphism, *i.e.* a trivialization of the flat bundle \mathcal{E} associated with λ and v is an isomorphism of flat bundles $\mathcal{E} \otimes E^- \rightarrow \mathcal{E} \otimes E^+$.

At the level of \tilde{V} , the trivialization φ corresponds to a Γ -equivariant isomorphism $\tilde{\varphi} : \tilde{V} \otimes N \rightarrow \tilde{V} \otimes P_\lambda$ and v to an unitary $u \in N \otimes M_n(\mathbb{C})$ intertwining $\lambda \otimes 1$ and $\lambda \otimes \alpha$ (P_λ is as in section 2.4).

Let then $\tilde{\mathcal{E}}$ denote the $C_0(\tilde{V}) \otimes Z_i$ module $\{f \in C_0(\tilde{V} \times [0, 1]) \otimes P_\lambda f(0) \in \tilde{\varphi}(C_0(\tilde{V}) \otimes 1)\}$. The pair $(\tilde{\mathcal{E}}, 0)$ defines a Kasparov bimodule in $\mathbf{E}^\Gamma(C_0(\tilde{V}), C_0(\tilde{V}) \otimes Z_i)$ joining $(C_0(\tilde{V}), 0)$ and $(C_0(\tilde{V}) \otimes P_\lambda, 0)$.

We may then perform the “ \diamond ” construction as in Theorem 2.7 using the isomorphism u to obtain an element in $z \in KK^\Gamma(C_0(\tilde{V}), C_0(\tilde{V}) \otimes Z_{i,i})$ whose image in $KK_{\mathbb{R}/\mathbb{Z}}^{1,\Gamma}(C_0(\tilde{V}), C_0(\tilde{V}))$ is $\rho_\alpha^{C_0(\tilde{V})}$. The element z is of course given by the Hilbert $C_0(\tilde{V}) \otimes Z_{i,i}$ -module

$$\tilde{\mathcal{E}} := \{f : [0, 1] \times \tilde{V} \rightarrow \mathbb{C}^n \otimes N : f(0, x) \in \mathbb{C}^n \times 1_N, \tilde{w}_v(x)f(1, x) \in \mathbb{C}^n \times 1_N, \text{ for every } x \in \tilde{V},$$

where $\tilde{w}_v = (\tilde{\varphi} \otimes 1)^{-1} \circ u^{-1} \circ (\tilde{\varphi} \otimes 1)$. As the left and right $C_0(\tilde{V})$ actions on $\tilde{\mathcal{E}}$ coincide, the element z is in the image of the forgetful map, $\mathcal{R}KK^\Gamma(\tilde{V}; C_0(\tilde{V}), C_0(\tilde{V}) \otimes Z_{i,i}) \rightarrow KK^\Gamma(C_0(\tilde{V}), C_0(\tilde{V}) \otimes Z_{i,i})$. Now $\tilde{\mathcal{E}}^\Gamma = \mathcal{E}$, and the result follows. \square

It follows then from Proposition 3.3 and Lemma 3.2 that we may deduce the class $[\alpha]_{APS}$ from our class $\rho_\alpha^{C_0(\tilde{V})}$.

3.2. If A is proper and free. Out of the example of the Mishchenko bundle of a compact manifold, we then obtain the following statements. Let Z be a locally finite CW complex realization for the classifying space $B\Gamma$ and $q : \tilde{Z} \rightarrow Z$ the corresponding covering with group Γ . Write $Z = \bigcup Z_n$, where Z_n is an increasing sequence of finite subcomplexes. Denote by N a II_1 -factor containig Γ in a trace preserving way, and $E_N = \tilde{Z} \times_\Gamma N$ the corresponding bundle over Z with fiber N .

- Lemma 3.4.** (1) *The restriction to Z_n of the bundle E_N is equal in $K_{\mathbb{R}}^0(Z_n)$ to the unit element (corresponding to the trivial bundle).*
- (2) *If \tilde{X} is a free, proper and cocompact Γ -space, then there is a II_1 -factor N with an embedding $\lambda : C^*\Gamma \rightarrow N$ such that $\text{tr}_N \circ \lambda = \mathbf{tr}$ and a unitary $w \in \mathcal{L}(C_0(\tilde{X}) \otimes N)$ intertwining the Γ -actions $\beta \otimes \text{id}_N$ and $\beta \otimes \lambda$ on $C_0(\tilde{X}) \otimes N$ where β is the action of Γ on $C_0(\tilde{X})$ by translation.*

Proof. (1) Since Z_n is a finite CW -complex, its K -theory is a finitely generated group. Therefore, using the Rosenberg-Schochet universal coefficient formula ([27, 26]), we find $KK_{\mathbb{R}}(C_0(Z_n), \mathbb{C}) = \text{Hom}(KK(\mathbb{C}, C_0(Z_n)); \mathbb{R})$. Using the Baum-Douglas theory (cf. [6]), we know that the K -homology of Z_n is generated by cycles given by elements $g_*(x)$ where g is a continuous map $g : V \rightarrow Z_n$, V is a compact manifold and $x \in K_*(V)$ is an element in the K -homology of V . But by Atiyah’s theorem for covering spaces ([2]), the pairing of E_N with such a K -homology element coincides with the index. It follows that the class of E_N is 1.

- (2) We have a continuous classifying map $f : X \rightarrow B\Gamma$ where $X = \tilde{X}$. Since X is compact its image sits in some Z_n and it follows that the bundle $f^*(E_N)$ defines the trivial element in $K_{\mathbb{R}}^0(X)$. Up to replacing N by $M_k(N)$, we may then assume that the bundle f^*E_N is trivial (as explained in [1, Prop. 5.1]). By definition of E_N , this means exactly that there exists a unitary $w \in \mathcal{L}(C_0(\tilde{X}) \otimes N)$ intertwining the Γ -actions $\beta \otimes \text{id}_N$ and $\beta \otimes \lambda$ on $C_0(\tilde{X}) \otimes N$. \square

We now recall the definition of free and proper Γ -algebras. Recall ([19]) that, if X is a locally compact space, a $C_0(X)$ algebra is a C^* -algebra A endowed with a morphism from

$C_0(X)$ into the center of the multiplier algebra of A and such that $A = C_0(X)A$. If X is a Γ space, a $C_0(X)$ – Γ -algebra A is an algebra endowed with compatible structures of $C_0(X)$ -algebra and Γ -algebra (*i.e.* the morphism $C_0(X) \rightarrow \mathcal{ZM}(A)$ is equivariant).

Let us recall facts about $C_0(X)$ -algebras (see [19, 20]):

- Properties 3.5.** (1) If A is a $C_0(X)$ -algebra, we may define for every open subset $U \subset X$ the $C_0(U)$ -algebra $A_U = C_0(U)A$. For every closed subset $F \subset X$, we put $A_F = A/A_{F^c}$, which is a $C_0(F)$ algebra. In particular, we have a fiber A_x for every point $x \in X$. Moreover, there is a natural evaluation map $a \mapsto a_x$ from A to A_x and for $a \in A$, we have $\|a\| = \sup\{\|a_x\|, x \in X\}$.
- (2) If A is a $C_0(X)$ -algebra and $f : Y \rightarrow X$ is a continuous map, we define a pull back $f^*(A)$ which is a $C_0(Y)$ -algebra. This pull back satisfies $f^*(A)_y = A_{f(y)}$ for all $y \in Y$. We will use the following facts:
- (a) If T is a locally compact space and $p : X \times T \rightarrow X$ is the projection then $p^*(A) = A \otimes C_0(T)$.
- (b) If $h : Y \rightarrow Y$ is a homeomorphism such that $f \circ h = f$, there is an automorphism $\theta^h : f^*(A) \rightarrow f^*(A)$ such that $(\theta^h(b))_y = b_{h(y)}$ for every $y \in Y$. Note that by property (1) this equality characterizes $\theta^h(b)$.

Definition 3.6. A Γ -algebra A is said to be *free* (*resp. proper*) if there exists a free (*resp. proper*) Γ -space \tilde{X} such that A is a $C_0(\tilde{X})$ – Γ -algebra.

Note that if A is free and proper, then there exist a free Γ -space \tilde{X}_1 and a proper Γ -space \tilde{X}_2 such that A is a $C_0(\tilde{X}_1)$ – Γ -algebra and a $C_0(\tilde{X}_2)$ – Γ -algebra. Then A is a $C_0(\tilde{X}_1 \times \tilde{X}_2)$ – Γ -algebra, and $\tilde{X}_1 \times \tilde{X}_2$ is free and proper.

We will next prove that every free and proper Γ -algebra is K -theoretically free and proper (Theorem 3.10).

Let us start with the cocompact case:

Proposition 3.7. *If \tilde{X} is a free, proper and cocompact Γ -space, then every $C_0(\tilde{X})$ – Γ -algebra satisfies property (KFP).*

Proof. Let A be a $C_0(\tilde{X})$ – Γ -algebra. Extending the action by left multiplication of $C_0(\tilde{X}) \otimes N$ on $A \otimes N$ to the multiplier algebra, we find a unitary $\hat{w} \in \mathcal{L}(A \otimes N)$ intertwining the actions $\hat{\beta} \otimes \text{id}_N$ and $\hat{\beta} \otimes \lambda$ of Γ . It follows that A satisfies property (KFP). \square

As above, let Z be a locally finite CW complex realization for the classifying space $B\Gamma$. Write $Z = \bigcup Z_n$, where Z_n is an increasing sequence of finite subcomplexes. Using local finiteness, we will also assume that Z_n is contained in the interior of Z_{n+1} .

Let \tilde{X} be a free and proper Γ -space and let A be a (separable) $C_0(\tilde{X})$ – Γ -algebra. Put $X = \tilde{X}/\Gamma$ and let $q : \tilde{X} \rightarrow X$ be the quotient. Let also $f : X \rightarrow Z$ be a classifying map.

Define $\tilde{X}_n = (f \circ q)^{-1}(\dot{Z}_n) \subset \tilde{X}$ and $\Omega = \bigcup_{n \in \mathbb{N}} \tilde{X}_n \times]n, +\infty[$.

Let $\mathcal{A} = C_0(\Omega)(A \otimes C_0(\mathbb{R}))$ and $A_n = C_0(X_n)A$. Let \mathfrak{A} denote the c_0 -sum $\bigoplus_{n \in \mathbb{N}} A_n$ and $j : \mathfrak{A} \rightarrow \mathfrak{A}$ the shift $(a_0, a_1, a_2, \dots) \mapsto (0, a_0, a_1, a_2, \dots)$.

Lemma 3.8. (1) *The Γ -algebras $A \otimes C_0(\mathbb{R})$ and \mathcal{A} are homotopy equivalent.*

(2) *The algebra \mathcal{A} identifies with the torus algebra $\mathcal{T}(j, \text{id}_{\mathfrak{A}})$ (see definition 1.2).*

Proof. (1) Of course $A \otimes C_0(\mathbb{R})$ and $A \otimes C_0(\mathbb{R}_+^*)$ are homotopy equivalent. Moreover let $h : Z \rightarrow \mathbb{R}_+$ be a (proper) map such that $h(z) > n$ if $z \notin Z_n$, and put $\hat{h} = h \circ f \circ q : X \rightarrow$

\mathbb{R}_+^* . The map $(x, t) \mapsto (x, t + \hat{h}(x))$ is a homeomorphism $\tilde{X} \times \mathbb{R} \rightarrow \tilde{X} \times \mathbb{R}$. It induces an equivariant automorphism of $A \otimes C_0(\mathbb{R})$ mapping $A \otimes C_0(\mathbb{R}_+^*)$ into \mathcal{A} which is a homotopy inverse of the inclusion $\mathcal{A} \rightarrow A \otimes C_0(\mathbb{R}_+^*)$.

- (2) An element in $A \otimes C_0(\mathbb{R}_+^*)$ is a map $\xi : \mathbb{R}_+^* \rightarrow A$. Such a ξ is in \mathcal{A} if and only if, for all $n \in \mathcal{V}$ and $t \in]n, n+1]$, we have $\xi(t) \in A_n$ for $n < t \leq n+1$. Associated to ξ is a map $\zeta : [0, 1] \rightarrow \mathfrak{A}$ defined by $\zeta(t)_n = \xi(n+t)$ ($\in A_n$). It is immediately seen that $\zeta \in \mathcal{T}(j, \text{id}_{\mathfrak{A}})$ and that $\xi \mapsto \zeta$ is an isomorphism. \square

Lemma 3.9. (1) *For every n , the algebra A_n satisfies property (KFP).*

- (2) *The algebra \mathfrak{A} satisfies property (KFP).*
 (3) *The algebra \mathcal{A} satisfies property (KFP).*
 (4) *The algebra A satisfies property (KFP).*

Proof. (1) Through the equivariant map $\tilde{X}_n \rightarrow \tilde{Z}_n$, the algebra A_n is a \tilde{Z}_n algebra. As \tilde{Z}_n is a free proper and cocompact Γ -space, the algebra A_n satisfies property (KFP) by prop.3.7.

- (2) For every Γ -algebra B the map $KK_{\mathbb{R}}^{\Gamma}(\mathfrak{A}, B) \prod_n KK_{\mathbb{R}}^{\Gamma}(A_n, B)$ is an isomorphism - the inverse map associates to a sequence (E_n, F_n) of bimodules (with F_n bounded) defining elements in $KK_{\mathbb{R}}^{\Gamma}(A_n, B)$ the class of $(\bigoplus E_n, F)$ where F is defined by $F((x_n)_n) = (F_n(x_n))_n$ (for $(x_n) \in \bigoplus_n E_n$) (see [26, 27]). Now $1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma}$ acts trivially on each copy $KK_{\mathbb{R}}^{\Gamma}(A_n, B)$, whence $1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma}$ acts as the 0 element in $KK_{\mathbb{R}}^{\Gamma}(\mathfrak{A}, \mathfrak{A})$.
 (3) Write $\mathcal{A} = \mathcal{T}(j, \text{id}_{\mathfrak{A}})$ and let $p : \mathcal{A} \rightarrow \mathfrak{A}$ and $h : S\mathfrak{A} \rightarrow \mathcal{A}$ be the associated map maps ($p : (a, f) \mapsto a$ and $h : f \mapsto (0, f)$). We have $p_*(1_{\mathcal{A}}^{\Gamma} \otimes_{\mathbb{C}} (1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma})) = p^*(1_{\mathfrak{A}}^{\Gamma} \otimes_{\mathbb{C}} (1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma})) = 0$ since \mathfrak{A} satisfies property (KFP).

By the torus exact sequence (prop. 1.2), there exists $y \in KK_{\mathbb{R}}^{\Gamma}(\mathfrak{A}, S\mathfrak{A})$ such that $1_{\mathcal{A}}^{\Gamma} \otimes_{\mathbb{C}} (1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma}) = h_* y$. As \mathfrak{A} satisfies property (KFP), we have $y \otimes_{S\mathfrak{A}} (1_{S\mathfrak{A}}^{\Gamma} \otimes_{\mathbb{C}} (1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma})) = 0$. As the Kasparov product over \mathbb{C} is commutative, we find

$$y \otimes_{S\mathfrak{A}} (1_{S\mathfrak{A}}^{\Gamma} \otimes_{\mathbb{C}} (1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma})) = (1_{\mathcal{A}}^{\Gamma} \otimes_{\mathbb{C}} (1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma})) \otimes_{\mathcal{A}} y$$

whence $(1_{\mathcal{A}}^{\Gamma} \otimes_{\mathbb{C}} (1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma})) \otimes_{\mathcal{A}} h_* y$, which yields

$$1_{\mathcal{A}}^{\Gamma} \otimes_{\mathbb{C}} ((1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma}) \otimes_{\mathbb{C}} (1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma})) = (1_{\mathcal{A}}^{\Gamma} \otimes_{\mathbb{C}} (1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma})) \otimes_{\mathcal{A}} (1_{\mathcal{A}}^{\Gamma} \otimes_{\mathbb{C}} (1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma})) = 0.$$

But $[\text{tr}]^{\Gamma}$ is idempotent in $KK_{\mathbb{R}}^{\Gamma}(\mathbb{C}, \mathbb{C})$, thus $(1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma})^2 = (1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma})$.

- (4) The Γ -algebra SA is homotopy equivalent to \mathcal{A} . It follows that SA satisfies property (KFP); thus $1_{SA}^{\Gamma} \otimes_{\mathbb{C}} (1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma}) = 0$. The Bott isomorphism $x \mapsto Sx$ from $KK_{\mathbb{R}}^{\Gamma}(C, D)$ to $KK_{\mathbb{R}}^{\Gamma}(SC, SD)$ maps $1_A^{\Gamma} \otimes_{\mathbb{C}} (1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma})$ to $1_{SA}^{\Gamma} \otimes_{\mathbb{C}} (1_{\mathbb{R}}^{\Gamma} - [\text{tr}]^{\Gamma}) = 0$. It follows that A satisfies property (KFP). \square

We have proved:

Theorem 3.10. *Every free and proper Γ -algebra satisfies property (KFP).* \square

As a corollary, every Γ -algebra which is KK -subequivalent to a proper and free algebra satisfies property (KFP).

3.3. If Γ satisfies (the KK^{Γ} - form of) the Baum-Connes conjecture. Let us say that Γ satisfies the KK^{Γ} -form of the Baum-Connes conjecture if there is a proper Γ -algebra \mathcal{Q} such that \mathbb{C} is KK^{Γ} -subequivalent to \mathcal{Q} . This of course implies that Γ is K -amenable ([10]). On the other hand, Higson and Kasparov proved that every A- T -menable group satisfies this property ([16, 17]).

If Γ satisfies the KK^Γ -form of the Baum-Connes conjecture every Γ -algebra is KK^Γ -subequivalent to a proper Γ -algebra (namely A is KK^Γ -subequivalent to $\mathcal{Q} \otimes A$).

- If Γ is torsion free. Then \mathcal{Q} is automatically free. Then every Γ -algebra satisfies property (KFP).
- In the general case. Assume A is free ⁽¹⁾, then $\mathcal{Q} \otimes A$ is free and proper and therefore satisfies property (KFP). It follows that every free Γ -algebra satisfies property (KFP).

3.4. If Γ has a γ element. Recall (cf. [19, 30]) that a γ -element for Γ is an element $\gamma \in KK^\Gamma(\mathbb{C}, \mathbb{C})$ such that:

- (1) There exists a proper Γ -algebra \mathcal{Q} and elements $D \in KK_\Gamma(\mathcal{Q}, \mathbb{C})$ and $\eta \in KK_\Gamma(\mathbb{C}, \mathcal{Q})$ such that $\gamma = \eta \otimes_{\mathcal{Q}} D$. D and η are respectively the so called *Dirac* and *dual Dirac* element;
- (2) for every proper Γ -algebra A , $1_A^\Gamma \otimes_{\mathbb{C}} \gamma = 1_A^\Gamma$.

Recall (cf. [30]), that if γ exists, it is unique.

- If Γ is torsion free and has a γ element, then the algebra \mathcal{Q} is free and proper, therefore it satisfies property (KFP). It follows that, for every finite dimensional unitary representation $\alpha : \Gamma \rightarrow U_n$, we can construct a canonical element $\rho_\alpha^\mathcal{Q} \in KK_{\mathbb{R}/\mathbb{Z}}^{1, \Gamma}(\mathcal{Q}, \mathcal{Q})$ and use the element η and D in order to define a canonical element $\rho_\alpha^\mathbb{C} = \eta \otimes_{\mathcal{Q}} \rho_\alpha^\mathcal{Q} \otimes_{\mathcal{Q}} D \in KK_{\mathbb{R}/\mathbb{Z}}^{1, \Gamma}(\mathbb{C}, \mathbb{C})$ in “the image of γ ”. Then for every Γ algebra, we construct $\rho_\alpha^A = 1_A^\Gamma \otimes_{\mathbb{C}} \rho_\alpha^\mathbb{C}$.
- In the same way as above, if Γ is no longer assumed to be torsion free, we may construct the element $\rho_\alpha^{\mathcal{Q} \otimes A}$ if A is free - and more generally if $\mathcal{Q} \otimes A$ is free. We then use the Dirac and dual Dirac elements to construct a canonical element $\rho_\alpha^A = \eta \otimes_{\mathcal{Q}} \rho_\alpha^{\mathcal{Q} \otimes A} \otimes_{\mathcal{Q}} D \in KK_{\mathbb{R}/\mathbb{Z}}^{1, \Gamma}(A, A)$ in “the image of γ ”.

3.5. If $\alpha(\Gamma)$ is torsion free. Put $\check{\Gamma} = \alpha(\Gamma)$ (with the discrete topology). Since $\check{\Gamma}$ is linear, it follows from [15] that it has a γ element. We may thus define an element $\rho_\alpha^\mathbb{C} \in KK_{\mathbb{R}/\mathbb{Z}}^{1, \check{\Gamma}}(\mathbb{C}, \mathbb{C})$ where $\check{\alpha}$ is the inclusion $\check{\Gamma} \subset U_n$. Then we can use the morphism $q : \Gamma \rightarrow \check{\Gamma}$ in order to define $\rho_\alpha^\mathbb{C} = q^* \rho_\alpha^\mathbb{C}$ (and then put, for every Γ -algebra $\rho_\alpha^A = 1_A^\Gamma \otimes_{\mathbb{C}} \rho_\alpha^\mathbb{C}$).

4. WEAK (KFP) PROPERTY

4.1. KK -theory elements with real coefficients associated with a trace on $C^*\Gamma$. As explained in Lemma 1.14, for every Γ -algebra A and every C^* -algebra B , we may identify $KK(A \rtimes \Gamma, B)$ with $KK^\Gamma(A, B)$ where B is endowed with the trivial action of B . Replacing in this formula B by $B \otimes M$ where M is a II_1 -factor and taking the inductive limit, we find for a Γ -algebra A and a C^* -algebra B endowed with a trivial Γ -action an identification $KK_\mathbb{R}^\Gamma(A, B) = KK_\mathbb{R}(A \rtimes \Gamma, B)$ - and in the same way, $KK_{\mathbb{R}/\mathbb{Z}}^\Gamma(A, B) = KK_{\mathbb{R}/\mathbb{Z}}(A \rtimes \Gamma, B)$.

Definition 4.1. Given a Γ -algebra A and a trace τ on $C^*\Gamma$ we have three equivalent ways to define the same element $[\delta_\tau^A] \in KK_\mathbb{R}(A \rtimes \Gamma, A \rtimes \Gamma)$:

- (1) We may put $[\delta_\tau^A] = \tau_*[\delta^A]$ where $\delta^A \in KK(A \rtimes \Gamma, A \rtimes \Gamma \otimes C^*\Gamma)$ is the class of the morphism δ^A and $\tau_* : KK(B, C \otimes C^*\Gamma) \rightarrow KK_\mathbb{R}(B, C)$ is the map associated with the trace τ constructed in Definition 1.7.

¹Note that we only need to assume that $\mathcal{Q} \otimes A$ is free, which is much weaker: it is just a statement on the torsion elements of Γ .

- (2) Starting with $[\tau] \in KK_{\mathbb{R}}(C^*\Gamma, \mathbb{C})$, we obtain (by tensoring with $A \rtimes \Gamma$) an element $(1_{A \rtimes \Gamma} \otimes [\tau]) \in KK_{\mathbb{R}}(A \rtimes \Gamma \otimes C^*\Gamma, A \rtimes \Gamma)$ and, with this notation, we have $[\delta_{\tau}^A] = (\delta^A)^*(1_{A \rtimes \Gamma} \otimes [\tau]) \in KK_{\mathbb{R}}(A \rtimes \Gamma, A \rtimes \Gamma)$.
- (3) Let as usual $[\tau]^{\Gamma} \in KK_{\mathbb{R}}^{\Gamma}(\mathbb{C}, \mathbb{C})$ be the element corresponding to $[\tau]$ via the identification of $KK_{\mathbb{R}}(C^*\Gamma, \mathbb{C})$ with $KK_{\mathbb{R}}^{\Gamma}(\mathbb{C}, \mathbb{C})$; the element $1_A^{\Gamma} \otimes [\tau]^{\Gamma} \in KK_{\mathbb{R}}^{\Gamma}(A, A)$ satisfies $[\delta_{\tau}^A] = j_{\Gamma}(1_A^{\Gamma} \otimes [\tau]^{\Gamma})$ where $j_{\Gamma} : KK_{\mathbb{R}}^{\Gamma}(A, A) \rightarrow KK_{\mathbb{R}}(A \rtimes \Gamma, A \rtimes \Gamma)$ is Kasparov's descent.

4.2. Weakly K-theoretically “free and proper” algebras. We now introduce a weakening of (KFP) property

Definition 4.2. Let Γ be a discrete group. Denote by \mathbf{tr} its group tracial state. We say that the Γ -algebra A satisfies the weak (KFP) property if $[\delta_{\mathbf{tr}}^A]$ is equal to the unit element $1_{A \rtimes \Gamma, \mathbb{R}}$ of the ring $KK_{\mathbb{R}}(A \rtimes \Gamma, A \rtimes \Gamma)$.

This conditions means that there is a II_1 -factor N with a morphism $\lambda : C^*\Gamma \rightarrow N$ such that $\text{tr}_N \circ \lambda = \mathbf{tr}$ (where tr_N is the normalized trace of N) and such that the maps $\delta_{\lambda}^A = (\text{id}_{A \rtimes \Gamma} \otimes \lambda) \circ \delta^A : A \rtimes \Gamma \rightarrow (A \rtimes \Gamma) \otimes N$ and $\iota^A = \text{id} \otimes 1_N : A \rtimes \Gamma \rightarrow (A \rtimes \Gamma) \otimes N$ define the same element in $KK(A \rtimes \Gamma, A \rtimes \Gamma \otimes N)$.

Let us make a few comments on this definition:

Properties 4.3. (1) If a Γ -algebra A satisfies property (KFP), it satisfies the weak (KFP) property.

Indeed, if $1_A^{\Gamma} \otimes [\mathbf{tr}]^{\Gamma} = 1_{A, \mathbb{R}}^{\Gamma}$, then $[\delta_{\mathbf{tr}}^A] = j_{\Gamma}(1_A^{\Gamma} \otimes [\mathbf{tr}]^{\Gamma}) = j_{\Gamma}(1_{A, \mathbb{R}}^{\Gamma}) = 1_{A \rtimes \Gamma, \mathbb{R}}$.

(2) If \mathbb{C} satisfies the weak (KFP) property, it satisfies property (KFP). Indeed, $\varepsilon_*[\delta_{\mathbf{tr}}^A] = [\mathbf{tr}] \in KK_{\mathbb{R}}(C^*\Gamma, \mathbb{C}) = KK_{\mathbb{R}}^{\Gamma}(\mathbb{C}, \mathbb{C})$.

(3) Let A, B be Γ -algebras. If A is KK^{Γ} -subequivalent to B and B satisfies the weak (KFP) property then so does A .

The assumption means that there exist $x \in KK^{\Gamma}(A, B)$ and $y \in KK^{\Gamma}(B, A)$ satisfying $x \otimes_B y = 1_A^{\Gamma}$. Note that $y \otimes_{\mathbb{C}} [\mathbf{tr}]^{\Gamma} = [\mathbf{tr}]^{\Gamma} \otimes_{\mathbb{C}} y = (1_B^{\Gamma} \otimes [\mathbf{tr}]^{\Gamma}) \otimes_B y$. Assume B satisfies the weak (KFP) property. Then $j_{\Gamma}(y \otimes_{\mathbb{C}} [\mathbf{tr}]^{\Gamma})$ is the image in $KK_{\mathbb{R}}(B \rtimes \Gamma, A \rtimes \Gamma)$ of $j_{\Gamma}(y)$, whence $[\delta_{\mathbf{tr}}^A] = j_{\Gamma}(x) \otimes_{B \rtimes \Gamma} j_{\Gamma}(y \otimes_{\mathbb{C}} [\mathbf{tr}]^{\Gamma}) = 1_{A \rtimes \Gamma, \mathbb{R}}$.

Remarks 4.4. (1) (see Remark 2.3) If A satisfies the weak (KFP) property, then $[\delta_{\tau}^A] = 1_{A \rtimes \Gamma, \mathbb{R}}$ for any tracial state τ .

(2) (see Remark 2.4) Given a von Neumann algebra N and a morphism $\lambda : C^*\Gamma \rightarrow N$ such that $\text{tr}_N \circ \lambda = \mathbf{tr}$ (where tr_N is the normalized trace of N), then λ factors through $C_{red}^*(\Gamma)$; it therefore defines a morphism $\widehat{\delta}_{\lambda}^A : A \rtimes_{red} \Gamma \rightarrow A \rtimes \Gamma \otimes N$. Let $q_{\Gamma} : A \rtimes \Gamma \rightarrow A \rtimes_{red} \Gamma$ be the natural morphism. If A satisfies the weak (KFP) property, then $q_{\Gamma}^*[\widehat{\delta}_{\lambda}^A] = 1_{A \rtimes \Gamma, \mathbb{R}}$ i.e the morphism $q_{\Gamma} : A \rtimes \Gamma \rightarrow A \rtimes_{red} \Gamma$ has a $KK_{\mathbb{R}}$ one sided inverse.

4.3. A construction involving coactions and torus algebras. Throughout this section, the following objects will be fixed: Γ is a discrete group, A is a Γ -algebra, α is a finite dimensional unitary representation of Γ . Moreover, we fix a II_1 -factor N and a morphism $\lambda : C^*\Gamma \rightarrow N$ such that $\text{tr}_N \circ \lambda = \mathbf{tr}$, where as above tr_N denotes the normalized trace of N and \mathbf{tr} is the group trace of Γ .

We fix two morphisms $\iota^A, \delta_{\lambda}^A : A \rtimes \Gamma \rightarrow A \rtimes \Gamma \otimes N$ where $\iota^A : x \mapsto x \otimes 1$ and δ_{λ}^A uses the “coaction” of Γ and is defined in Notation 4.5 below.

We now construct an element $\sigma_\alpha^A \in KK_{\mathbb{R}/\mathbb{Z}}^1(\mathcal{T}, A \rtimes \Gamma)$ where $\mathcal{T} = \mathcal{T}(\iota^A, \delta_\lambda^A)$ is the corresponding torus algebra. This will be an important ingredient in our construction of the weaker ρ invariant (Definition 4.11).

4.3.1. A morphism between torus algebras. There exists a II_1 -factor M containing N with a morphism $C(U_n) \rtimes \Gamma \rightarrow M$ extending $\lambda : C^*\Gamma \rightarrow N \subset M$. Indeed we can take as M (a II_1 -factor containing) the free product of N with $L^\infty(U_n) \rtimes \Gamma$ with amalgamation over the group von Neumann algebra of Γ .

It follows, thanks to Lemma 2.5, that there exists a unitary $u \in M_n(\mathbb{C}) \otimes M$ satisfying $(\alpha(g) \otimes \lambda(g))u = u(1 \otimes \lambda(g)) \in M_n(\mathbb{C}) \otimes M$.

Notation 4.5. Given a unital C^* -algebra B and a unitary representation $\pi : \Gamma \rightarrow B$ - in other words a morphism $\pi : C^*\Gamma \rightarrow B$ we will put

$$\delta_\pi^A := (\text{id}_{A \rtimes \Gamma} \otimes \pi) \circ \delta^A : A \rtimes \Gamma \rightarrow (A \rtimes \Gamma) \otimes B.$$

We denote by $\alpha : C^*\Gamma \rightarrow M_n(\mathbb{C})$ the morphism corresponding to $\alpha : \Gamma \rightarrow U_n$. We obtain a morphism $\delta_\alpha^A = (\text{id}_{A \rtimes \Gamma} \otimes \alpha) \circ \delta^A : A \rtimes \Gamma \rightarrow (A \rtimes \Gamma) \otimes M_n$. Denote by $[\delta_\alpha^A] \in KK(A \rtimes \Gamma, A \rtimes \Gamma)$ the associated KK -class.

Proposition 4.6. *Let M be a II_1 -factor containing N and let $u \in M_n(\mathbb{C}) \otimes M$ be a unitary satisfying $(\alpha(g) \otimes \lambda(g))u = u(1 \otimes \lambda(g)) \in M_n(\mathbb{C}) \otimes M$.*

Denote by $j_n^A : A \rtimes \Gamma \rightarrow A \rtimes \Gamma \otimes M_n(\mathbb{C})$ the morphism given by $x \mapsto x \otimes 1_{M_n(\mathbb{C})}$ and $\iota^A : A \rtimes \Gamma \rightarrow A \rtimes \Gamma \otimes N$ the one given by $x \mapsto x \otimes 1_N$. Let $i_n : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \otimes M$ the map defined by $i_n(x) = x \otimes 1_M$ and $\widehat{i_n} = \text{Ad}_u \circ i_n$.

- (1) *There is a morphism $\Delta_\alpha^u : \mathcal{T}(\iota^A, \delta_\lambda^A) \rightarrow A \rtimes \Gamma \otimes Z(i_n, \widehat{i_n}) = Z(\text{id}_{A \rtimes \Gamma} \otimes i_n, \text{id}_{A \rtimes \Gamma} \otimes \widehat{i_n})$ such that $\Delta_\alpha^u(a, f) = (\delta_\alpha^A(a), j_n^A(a), g)$ with*

$$g(t) = \begin{cases} (\delta_\alpha^A \otimes i_{M,N})(f(2t)) & \text{for } t \leq 1/2 \\ (1_{A \rtimes \Gamma} \otimes u)(j_n^A \otimes i_{M,N})f(2-2t)(1_{A \rtimes \Gamma} \otimes u^*) & \text{for } t \geq 1/2. \end{cases}$$

- (2) *Let $p : \mathcal{T}(\iota^A, \delta_\lambda^A) \rightarrow A \rtimes \Gamma$ the projection $p : (a, f) \mapsto a$. With respect to the natural evaluation maps $p_0, p_1 : A \rtimes \Gamma \otimes Z(i_n, \widehat{i_n}) \otimes M_n$, we have*

$$\begin{aligned} p_0 \circ \Delta_\alpha^u &= \delta_\alpha^A \circ p, & p_0 : (a_0, a_1, f) &\mapsto a_0 \\ p_1 \circ \Delta_\alpha^u &= j_n^A \circ p, & p_1 : (a_0, a_1, f) &\mapsto a_1 \end{aligned}$$

- (3) *Furthermore $[\Delta_\alpha^u \circ h] = \hat{h}_*([[\delta_\alpha^A] - [j_n^A]] \otimes [Si_{M,N}]) \in KK(A \rtimes \Gamma \otimes SN, Z(i_n, \widehat{i_n}))$ where $Si_{M,N}$ is the inclusion $SN \hookrightarrow SM$ and where $h : A \rtimes \Gamma \otimes SN \rightarrow \mathcal{T}(\iota^A, \delta_\lambda^A)$ and $\hat{h} : A \rtimes \Gamma \otimes M_n(\mathbb{C}) \otimes SM \rightarrow A \rtimes \Gamma \otimes Z(i_n, \widehat{i_n})$ denote the inclusions $f \mapsto (0, f)$ and $f \mapsto (0, 0, f)$.*

Proof. (1) The morphism is well defined. (2) is immediate. (3) Obviously $\Delta_\alpha^u(A \rtimes \Gamma \otimes SN) \subset A \rtimes \Gamma \otimes M_n(\mathbb{C}) \otimes SM$. We just have to compute the KK -class of the morphism $\check{\Delta}_\alpha^u : A \rtimes \Gamma \otimes SN \rightarrow A \rtimes \Gamma \otimes M_n(\mathbb{C}) \otimes SM$ induced by Δ_α^u . Since the action of Ad_{u^*} is trivial in K -theory, we are computing the class of the sum $g_1 + g_2$ where:

$$g_1(t) = \begin{cases} (\delta_\alpha^A \otimes i_{M,N})f(2t) & \text{for } t \leq 1/2 \\ 0 & \text{for } t \geq 1/2. \end{cases}$$

and

$$g_2(t) = \begin{cases} 0 & \text{for } t \leq 1/2 \\ (j_n^A \otimes i_{M,N})f(2-2t) & \text{for } t \geq 1/2. \end{cases}$$

Both are morphisms from $A \rtimes \Gamma \otimes SN \longrightarrow A \rtimes \Gamma \otimes M_n(\mathbb{C}) \otimes SN$ composed with the inclusion $i_{M,N}$. Since the ranges of g_1 and g_2 are orthogonal, the sum as morphisms correspond to the sum of the KK -classes. Now the class of g_2 is the class of the morphism $S\delta_\alpha^A : A \rtimes \Gamma \otimes SN \longrightarrow A \otimes \Gamma \otimes SN$ induced on the suspension and $[S\delta_\alpha^A] = 1_{SN} \otimes [\delta_\alpha^A]$. The class of g_1 is easily seen to be the opposite of the class of the morphism Sj_n^A , for which again $[Sj_n^A] = 1_{SN} \otimes [j_n^A]$. \square

In Remark 1.3, we constructed (using u) a bimodule yielding a Morita equivalence between the torus algebras $Z(i_n, i_n)$ and $Z(i_n, \widehat{i_n})$. The KK -class of this bimodule is an element in $KK(Z(i_n, \widehat{i_n}), Z(i_n, i_n))$. We denote by $[E_u]$ its image as a class in $KK(Z(i_n, \widehat{i_n}), Z(i, i))$, where we use the fact that $Z(i_n, i_n) = M_n \otimes Z_{i,i}$, where i is the inclusion $i : \mathbb{C} \rightarrow N$.

Define

$$\psi_\alpha^M := [\Delta_\alpha^u] \otimes [E_u] \in KK(\mathcal{T}, A \rtimes \Gamma \otimes Z_{i,i}). \quad (4.1)$$

Proposition 4.7. *Denote by $\tilde{h} : A \rtimes \Gamma \otimes M_n(\mathbb{C}) \otimes SM \rightarrow A \rtimes \Gamma \otimes M_n \otimes Z(i, i)$ the inclusion $f \mapsto (0, 0, f)$, and let $p_0, p_1 : A \rtimes \Gamma \otimes Z_{i,i} \rightarrow A \rtimes \Gamma$ be the natural evaluation maps.*

(1) *We have*

$$(p_0)_*(\psi_\alpha^M) = [\delta_\alpha^A \circ p], \quad (4.2)$$

$$(p_1)_*(\psi_\alpha^M) = [j_n^A \circ p]$$

$$h^*(\psi_\alpha^M) = \tilde{h}_*([\delta_\alpha^A] - [j_n^A]) \otimes [S i_{M,N}]. \quad (4.3)$$

(2) ψ_α^M *does not depend on* u ;

(3) ψ_α^M *behaves in a natural way with respect to inclusions $i_{\overline{M},M} : M \hookrightarrow \overline{M}$ of II_1 -factors, namely*

$$j_{\overline{M},M}(\psi_\alpha^M) = \psi_\alpha^{\overline{M}}$$

where $j_{\overline{M},M}$ is the map induced by $i_{\overline{M},M}$ on the cylinders.

Proof. (1) The equality follows directly by Proposition 4.6 and the first Remark 1.3.

(2) This follows from the connectedness of the group of unitaries of a von Neumann algebra. In fact, another unitary u' in $M_n(\mathbb{C}) \otimes M$ satisfying $(\alpha_g \otimes \lambda_g)u' = u'(1 \otimes \lambda_g)$ is of the form uv , where v is a unitary of the von Neumann algebra $(M_n(\mathbb{C}) \otimes M) \cap \{\lambda_g, g \in \Gamma\}'$.

(3) By part (2), we can construct $\psi_\alpha^{\overline{M}}$ using the unitary $\bar{u} = (\text{id}_{M_n(\mathbb{C})} \otimes i_{\overline{M},M})(u)$.

Let $\bar{i} : \mathbb{C} \rightarrow \overline{M}$ be the inclusion in \overline{M} , and $j_{\overline{M},M} : Z(i, i) \rightarrow Z(\bar{i}, \bar{i})$ the induced map on the cylinders. In this way, the Morita equivalence $[E_{\bar{u}}]$ used in the construction is compatible with the inclusion of cylinders. \square

Definition 4.8. We denote by σ_α^A the element

$$\sigma_\alpha^A = \vartheta(\psi_\alpha^M) \in KK_{\mathbb{R}/\mathbb{Z}}^1(\mathcal{T}(\iota^A, \delta_\lambda^A), A \rtimes \Gamma)$$

where $\vartheta : KK(\mathcal{T}(\iota^A, \delta_\lambda^A), A \rtimes \Gamma \otimes Z_{i,i}) \rightarrow KK_{\mathbb{R}/\mathbb{Z}}^1(\mathcal{T}(\iota^A, \delta_\lambda^A), A \rtimes \Gamma)$ is the morphism described in Remark 1.11.

Proposition 4.9. (1) *The class σ_α^A doesn't depend on the choices involved in this construction.*

(2) *The image $\partial(\sigma_\alpha^A) \in KK(\mathcal{T}(\iota^A, \delta_\lambda^A), A \rtimes \Gamma)$ through the connecting map of the Bockstein exact sequence is $\partial(\sigma_\alpha^A) = p^*([\delta_\alpha^A] - n1_{A \rtimes \Gamma})$.*

(3) *The image $h^*(\sigma_\alpha^A) \in KK_{\mathbb{R}/\mathbb{Z}}^1(A \rtimes \Gamma \otimes SN, A \rtimes \Gamma)$ is the image of $([\delta_\alpha^A] - n1_{A \rtimes \Gamma}) \otimes 1_{SN} \in KK(A \rtimes \Gamma \otimes SN, A \rtimes \Gamma \otimes SN) = KK^1(A \rtimes \Gamma \otimes SN, A \rtimes \Gamma \otimes N)$ through the composition $KK^1(A \rtimes \Gamma \otimes SN, A \rtimes \Gamma \otimes N) \longrightarrow KK_{\mathbb{R}}^1(A \rtimes \Gamma \otimes SN, A \rtimes \Gamma) \longrightarrow KK_{\mathbb{R}/\mathbb{Z}}^1(A \rtimes \Gamma \otimes SN, A \rtimes \Gamma)$.*

Proof. (1) We know that ψ_α^M does not depend on the choice of u . Furthermore, it behaves well under trace preserving inclusions of II_1 -factors. Since $KK_{\mathbb{R}/\mathbb{Z}}^1(\cdot, \cdot)$ is defined by the limit (1.2) with respect to all these inclusions, every choice of M defines the same class.

(2) It follows from Remark 1.11 and Proposition 4.6, in fact: $\partial(\sigma_\alpha^M) = (\partial \circ \vartheta)(\psi_\alpha^M) = [\delta_\alpha^A \circ p] - [j_n^A \circ p] = p^*([\delta_\alpha^A] - 1_{A \rtimes \Gamma})$, since $[j_n^A] = n1_{A \rtimes \Gamma}$.

(3) It is a consequence of (4.3), because the inclusion $SN \hookrightarrow C_i$ induces the change of coefficients $\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$. \square

Remark 4.10. Let $N_1 \subset N$ be a sub-factor of N containing $\lambda(\Gamma)$. The torus algebra $\mathcal{T}(\iota^A, \delta_\lambda^A)$ associated with N_1 is a subalgebra to the one associated with N . Obviously, the element σ_α^A associated with N_1 is just the restriction of the one of N .

4.4. Construction of $\hat{\rho}_\alpha^A$. Let A be a Γ -algebra satisfying the weak (KFP) property, and α a finite dimensional unitary representation of Γ .

By Remark 4.4, $[\delta_\alpha^A]_{\mathbb{R}} = n1_{A \rtimes \Gamma}^{\mathbb{R}}$ where $1_{A \rtimes \Gamma}^{\mathbb{R}}$ denotes the unit element of the ring $KK_{\mathbb{R}}(A \rtimes \Gamma, A \rtimes \Gamma)$. Using the Bockstein change of coefficients exact sequence, it follows that there exists $z \in KK_{\mathbb{R}/\mathbb{Z}}^1(A \rtimes \Gamma, A \rtimes \Gamma)$ whose image in $KK(A \rtimes \Gamma, A \rtimes \Gamma)$ is $[\delta_\alpha^A] - n1_{A \rtimes \Gamma}$.

We now construct a class $\hat{\rho}_\alpha^A \in KK_{\mathbb{R}/\mathbb{Z}}^1(A \rtimes \Gamma, A \rtimes \Gamma)$ whose image in $KK(A \rtimes \Gamma, A \rtimes \Gamma)$ is $[\delta_\alpha^A] - n1_{A \rtimes \Gamma}$.

The important fact in our construction, as for the one given in Definition 2.8 for the stronger context, is that the element $\hat{\rho}_\alpha^A$ is *independent of all choices*.

Using the torus $\mathcal{T}(\iota^A, \delta_\lambda^A)$ and the corresponding exact sequence (see Section 1.2), this means exactly that there exists a class $y \in KK(A \rtimes \Gamma, \mathcal{T}(\iota^A, \delta_\lambda^A))$ with $p_*(y) = 1_{A \rtimes \Gamma}$ where p is the projection $p: \mathcal{T}(\iota^A, \delta_\lambda^A) \longrightarrow A \rtimes \Gamma$.

Definition 4.11. We denote by $\hat{\rho}_\alpha^A \in KK_{\mathbb{R}/\mathbb{Z}}^1(A \rtimes \Gamma, A \rtimes \Gamma)$ the product $y \otimes_{\mathcal{T}(\iota^A, \delta_\lambda^A)} \sigma_\alpha^A$.

Theorem 4.12. (1) *The image of $\hat{\rho}_\alpha^A$ under the connecting map of the Bockstein change of coefficients exact sequence $KK_{\mathbb{R}/\mathbb{Z}}^1(A \rtimes \Gamma, A \rtimes \Gamma) \rightarrow KK(A \rtimes \Gamma, A \rtimes \Gamma)$ is $[\delta_\alpha^A] - n1_{A \rtimes \Gamma}$.*

(2) *The class $\hat{\rho}_\alpha^A$ does not depend on the choices of N and y involved in its construction.*

Proof. (1) We have $\partial \hat{\rho}_\alpha^A = y \otimes_{\mathcal{T}(\iota^A, \delta_\lambda^A)} \partial(\sigma_\alpha^A) = y \otimes_{\mathcal{T}(\iota^A, \delta_\lambda^A)} p^*([\delta_\alpha^A] - n1_{A \rtimes \Gamma})$ by Proposition 4.9. Whence $\partial \hat{\rho}_\alpha^A = p_* y \otimes_{A \rtimes \Gamma} ([\delta_\alpha^A] - n1_{A \rtimes \Gamma}) = [\delta_\alpha^A] - n1_{A \rtimes \Gamma}$ since $p_* y = 1_{A \rtimes \Gamma}$.

(2) Fix first N . We show that if $z \in KK(A \rtimes \Gamma, \mathcal{T}(\iota^A, \delta_\lambda^A))$ satisfies $p_*(z) = 0$ then $z \otimes_{\mathcal{T}(\delta_\lambda^A, \iota^A)} \sigma_\alpha^A = 0$. Using the torus exact sequence, it is enough to show that for every $z \in KK(A \rtimes \Gamma, A \rtimes \Gamma \otimes SN)$ we have $h_*(z) \otimes_{\mathcal{T}(\iota^A, \delta_\lambda^A)} \sigma_\alpha^A = z \otimes_{A \rtimes \Gamma \otimes SN} h^*(\sigma_\alpha^A) = 0$. But, as A satisfies the weak (KFP) property, $[\delta_\alpha^A] - n1_{A \rtimes \Gamma}$ is the 0 element in

$KK_{\mathbb{R}}(A \rtimes \Gamma, A \rtimes \Gamma)$ therefore, by Proposition 4.9, $h^*(\sigma_{\alpha}^A) = 0$. This shows that, given N , $\hat{\rho}_{\alpha}^A$ is independent on y .

Let N_1 be a II_1 -factor containing N . Denote by $j : N \rightarrow N_1$ the inclusion. Put $\lambda_1 = j \circ \lambda : C^*\Gamma \rightarrow N_1$, $\iota_A^1 = (\text{id}_{A \rtimes \Gamma} \otimes j) : A \rtimes \Gamma \rightarrow A \rtimes \Gamma \otimes N_1$ and $p_1 : \mathcal{T}(\iota_A^1, \delta_{\lambda_1}^A) \rightarrow A \rtimes \Gamma$ the morphism $(a, f) \mapsto a$. Put $y_1 = j_*(y) \in \mathcal{T}(\iota_A^1, \delta_{\lambda_1}^A)$. It follows from remark 4.10 that (N_1, y_1) and (N, y) define the same element $\hat{\rho}_{\alpha}^A$. Using amalgamated free products, we then see that $\hat{\rho}_{\alpha}^A$ is also independent on N . \square

Remark 4.13. If A satisfies property (KFP), it is in fact quite easy to compare the two constructions of ρ_{α}^A of Definition 2.8 and the one we just constructed, by showing that $\hat{\rho}_{\alpha}^A = j_{\Gamma}(\rho_{\alpha}^A)$, where $j_{\Gamma} : KK_{\mathbb{R}/\mathbb{Z}}^{\Gamma, 1}(A, A) \rightarrow KK_{\mathbb{R}/\mathbb{Z}}^1(A \rtimes \Gamma, A \rtimes \Gamma)$ is Kasparov's descent morphism.

Indeed, the Kasparov descent of a Kasparov bimodule $(E, F) \in \mathbf{E}^{\Gamma}(A, A \otimes Z_i)$ joining $(A, 0)$ and $(A \otimes P_{\lambda})$ is an element $y \in KK(A \rtimes \Gamma, \mathcal{T}(\iota_A^A, \delta_{\lambda}^A))$ as above; the gluing construction “ \diamond ” corresponds to the bimodule ψ_{α}^M . It follows that the image by j_{Γ} of (the class of) the bimodule

$$(E \otimes V_{\alpha}, F \otimes 1) \diamond_{\text{id}_A \otimes u} (E \otimes V_n, F \otimes 1)$$

is $y \otimes_{\mathcal{T}(\iota_A^A, \delta_{\lambda}^A)} \psi_{\alpha}^M$. Taking the images in $KK_{\mathbb{R}/\mathbb{Z}}^1$, the result follows.

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